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Non-Euclidean Elements

Intrinsic finite element methods in hyperbolic geometry

Karl Olav Tyssvang

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Karl Olav Tyssvang

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Supervisor:
Snorre H. Christiansen

Abstract

We first present an elementary construction of the hyperbolic spaces and give an overview of Galerkin methods. We then use these ideas to construct two coordinate independent finite element methods with interesting properties. In chapter 6 we present a metric dependent framework for the virtual element method, together with an implementation. We then construct coordinate independent polynomial-like spaces, which culminate in the construction of an intrinsic virtual element method. To the best of my knowledge, the H^1-projection which facilitates the final construction is novel.

Contents

Preface

Chapter 3 and 4 can be read with little prerequisites, after which chapter 5 is a natural continuation. Chapter 6 stands out as more exploratory and treads on uncharted territory. Because of this, chapter 6 is more targeted at those familiar with differential geometry and element methods.

We have implemented nearly all of the discretizations detailed in this master thesis. The code is available at *https://github.com/MolavyMakemake/Element-Method-Implementations*. Under the folder "meshgen" is the source code for the mesh generation software, which was implemented in C++ using the libraries *OpenGL* and *delaunator-cpp*[1]. Under the folder "hyperbolic" are the element method implementations, figures, and convergence data. The element methods were implemented in Python using the libraries *NumPy*, *SciPy*, and *matplotlib*.

Thank you to the University of Oslo. A special thank you to my advisor, Snorre H. Christiansen.

[1] https://github.com/delfrrr/delaunator-cpp/tree/master

1. Introduction

When faced with a computational problem a key strategy is to identify underlying structures. Building algorithms around such structures often results in improved convergence and stability. For example, symplectic methods take advantage of the analytic properties of Hamiltonian systems to allow for larger scale and more stable numerical schemes. A personal favorite example is leapfrog integration for the n-body problem [HLW03]. In the setting of finite elements, especially for problems formulated in terms of differential forms, the discrete de Rham complex is of great importance [AFW07]. Taking the de Rham complex as a starting point one can systematically construct the common finite element methods [Hip99], a lot of which are non-trivial, e.g. the Raviart-Thomas elements. This philosophy is at the heart of the field of *structure preserving discritizations* and will be our motivation throughout this master thesis.

The ultimate goal is to explore the construction of coordinate-independent (intrinsic) discretizations of PDE on Riemannian manifolds. Regge calculus [Chr11], and finite element systems based on harmonic extensions [CG16], are examples of such methods. There has recently been an interest in maintaining intrinsic properties for numerical schemes in mechanics [Dzi+24; BFP21], but the topic has received comparatively little attention in the non-Euclidean setting.

To enable implementation and comparison of methods we will restrict ourselves to the hyperbolic spaces, mostly the hyperbolic plane. The thesis is laid out as follows: In chapter 3 we construct the hyperbolic spaces as well as the models we will be using. In chapter 4 we describe how standard FEM can be used to solve problems on these geometries. In chapter 5 and 6 we present some intrinsic methods and compare them to the methods given in chapter 4, as well as describe some of the limitations of working in hyperbolic geometry.

2. Preliminaries

Definition 2.1. A topological space M is said to be a *smooth n-manifold* if it is Hausdorff, second countable, and there exists a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathscr{A}}$ such that

- $U_\alpha \subset \mathbb{R}^n$ is open and $\phi_\alpha : U_\alpha \to M$ is a homeomorphism for all $\alpha \in \mathscr{A}$;
- $\bigcup_{\alpha\in\mathscr{A}} \phi_\alpha(U_\alpha) = M$;
- $\phi_\beta^{-1} \circ \phi_\alpha$ is a diffeomorphism $\phi_\alpha^{-1}(\phi_\beta(U_\beta)) \to \phi_\beta^{-1}(\phi_\alpha(U_\alpha))$ for all $\alpha, \beta \in \mathscr{A}$.

Then $\mathfrak{U}$ is called an *atlas*, $(U, \phi) \in \mathfrak{U}$ is called a *chart*, and M is said to be of dimension n. The smoothness comes from the last requirement. If $\mathfrak{U}$ is a smooth atlas for M then it is contained in a unique *maximal atlas* $\mathfrak{M}$, and formally the manifold is the tuple $(M, \mathfrak{M})$. If M is of dimension n and of dimension m then $n = m$. For an introduction to manifolds we refer to [Tu11].

Definition 2.2. For a smooth manifold M we denote

- the set of smooth scalar functions by $C^\infty(M)$;
- for $p \in M$, the tangent space by $\mathrm{T_pM}$;
- the set of smooth vector fields by $\mathscr{X}(M)$;
- the set of smooth k-forms by $\Lambda^k M$.

On a coordinate chart we will denote the inherited basis for $\mathrm{T_pM}$ by $\partial_i|_p$, respectively the inherited basis for $(\mathrm{T_pM})^\star$ by $dx^i|_p$. Then ∂_i, respectively dx^i, are defined pointwise on the entire coordinate chart.

Definition 2.3. A *Riemannian manifold* (M, g) is a smooth manifold M together with a smooth *Riemannian metric* g, i.e. a function g s.t. $g|_p$ is an inner product on $\mathrm{T_pM}$ and $g(X, Y)$ is smooth for all $X, Y \in \mathscr{X}(M)$.

Definition 2.4. Given smooth manifolds M, N and a smooth map $F : M \to N$ we denote the pushforward $\mathscr{X}(M) \to \mathscr{X}(N)$ by $F_\star$.

Definition 2.5. A smooth map $F : (M, g) \to (N, h)$ between Riemannian manifolds is said to be an isometry if it is a diffeomorphism and $h(F_\star X, F_\star Y) = g(X, Y)$ for all $X, Y \in \mathscr{X}(M)$.

Going forward we will assume all manifolds are equipped with an inner product and use the notation $\langle X, Y\rangle := g(X, Y)$. The following is an overview of common constructions of Riemannian geometry. We refer to [Lee19] for a detailed presentation.

On a coordinate chart we will think of g as a matrix defined pointwise by $g_{ij} := \langle\partial_i, \partial_j\rangle$, and we will denote its inverse by $g^{ij} := (g^{-1})_{ij}$. Given $X = \sum X_i\partial_i$ and $Y = \sum Y_i\partial_i$ this allows us to write

$$\langle X, Y\rangle = \sum_{i,j} X_i g_{ij} Y_j.$$

There is a canonical isomorphism $\flat : \mathscr{X}(M) \to \Lambda^1 M$ given by $X^\flat = \langle X, \cdot\rangle$; its inverse, denoted by $\sharp$, is guaranteed by Riesz representation theorem. This is referred to as the *musical isomorphism*. Using the musical isomorphism we can extend g to an inner product on $\Lambda^1 M$ by setting $\langle\alpha, \beta\rangle = \langle\alpha^\sharp, \beta^\sharp\rangle$.

On a coordinate chart, given $\alpha = \sum a_i dx^i$, we have that $\alpha^\sharp = \sum_{i,j} a_i g^{ij}\partial_j$. Indeed,

$$\langle\sum_{i,j} a_i g^{ij}\partial_j, \partial_k\rangle = \sum_{i,j} a_i g^{ij} g_{jk} = a_k = \alpha(\partial_k).$$

If $\beta = \sum_i b_i dx^i$ this allows us to write their product explicitly as

$$\langle\alpha, \beta\rangle = \sum_{i,j} a_i g^{ij} b_j.$$

In general, given $\alpha, \beta \in \Lambda^k M$, on a coordinate chart we may write uniquely

$$\alpha = \sum_{I\in\mathscr{I}^k} a_I dx^I, \quad \beta = \sum_{I\in\mathscr{I}^k} b_I dx^I,$$

where $\mathscr{I}^k$ is the set of strictly increasing multi-indices of length k such that $I_i \le \dim M$ and $dx^I = dx^{I_1} \wedge \cdots \wedge dx^{I_k}$. The Gram determinant extends g to an inner product on $\Lambda^k M$ given pointwise by

$$\langle\alpha, \beta\rangle := \sum_{I,J\in\mathscr{I}^k} a_I b_J \det\Big(\langle dx^{I_i}, dx^{J_i}\rangle_{i,j}\Big).$$

The extension of the inner product lets us define the volume form μ_g as the unique positively oriented n-form such that $\langle\mu_g, \mu_g\rangle \equiv 1$. On a coordinate chart the volume form is given explicitly by $\mu_g = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$.

Definition 2.6. The *Hodge star operator*, denoted $\star$, is the unique homomorphism $\Lambda^k M \to \Lambda^{n-k} M$ such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \mu_g \qquad \forall \alpha, \beta \in \Lambda^k M.$$

Proposition 2.7. $\star^{-1}\omega = (-1)^{k(n-k)} \star\omega$ *for all* $\omega \in \Lambda^k M$.

Definition 2.8. Given $f \in C^\infty(M)$ and $X \in \mathscr{X}(M)$, the Hodge star lets us define the common operators of vector calculus as follows:

$$\begin{aligned} \Delta f &:= \star d \star df, \\ \operatorname{div} X &:= \star d \star X^\flat, \\ \operatorname{curl} X &:= \star d X^\flat. \end{aligned}$$

On a coordinate chart, with $X = \sum X_i \partial_i$, the local formulations of Δ and div are given by

$$\begin{aligned} \Delta f &= \frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i (\sqrt{|g|} \, g^{ij} \partial_j f), \qquad (1) \\ \operatorname{div} X &= \frac{1}{\sqrt{|g|}} \sum_i \partial_i (\sqrt{|g|} X_i). \end{aligned}$$

Definition 2.9. A curve $\gamma : [0, T] \to M$ is a geodesic if its length $\int_0^T \|\gamma'\|$ is minimal in the space of curves from $\gamma(0)$ to $\gamma(T)$. It is said to be a geodesic parametrization if $\|\gamma'\| \equiv 1$. We will also refer to the image of γ as a geodesic.

Definition 2.10. The integral of $f \in C^\infty(M)$ is defined as

$$\int_M f := \int_M f \mu_g.$$

Theorem 2.11 (Stokes). *If the boundary of M is Lipschitz-continuous then*

$$\int_{\partial M} \alpha = \int_M d\alpha$$

for all $\alpha \in \Lambda^{n-1}(M)$.

Corollary 2.12. *For all* $f, g \in C^\infty(M)$

$$\int_{\partial M} f \star dg = \int_M \langle df, dg \rangle + \int_M f \Delta g. \qquad (2)$$

Theorem 2.13 (Lax-Milgram). *Let H be a Hilbert space. A continuous bilinear form a is said to be coercive if there exits $\alpha > 0$ such that $a(v, v) \geq \alpha \|v\|^2$ for all $v \in H$. If a is coercive and $l : H \to \mathbb{R}$ is continuous and linear there exists a unique $u \in H$ such that $a(u, v) = l(v)$ for all $v \in H$.*

To avoid excessive notation we will assume all computations are done with respect to the underlying metric. In the case that we need to refer to computations in local coordinates, say (U, ϕ), we will say that we are working over the coordinates of U and denote by a subscript 0. For example

$$\int_{\phi(U)} f = \int_{\phi(U)} f \mu_g = \int_U f \sqrt{|g|} \, \mu_0.$$

We make an exception for the Euclidean norm of coordinates since it will shows up frequently in local formulations, i.e. we maintain the familiar notation $\|x\|$ for $x \in U$. Later when we are working in global coordinates we will be a bit relaxed in integral notation in the sense that we may write $\int_U f$ in place of $\int_{\phi(U)} f$.

We reserve $e^1, \cdots, e^n$ to denote the common orthonormal basis for $\mathbb{R}^n$. We denote by $\mathbb{P}_k(U)$ the space of k-order polynomials over U, and by $\mathbb{P}_k^{\text{hom}}(U)$ the space of k-order homogeneous polynomials.

3. Hyperbolic space

Gaussian curvature is a geometric property of 2-manifolds which at each point relates to the growth in area of nearby space. In the study of 2-manifolds isometrically embedded in $\mathbb{R}^3$ this is realized as the determinant of the differential of the normal field, which was famously shown to be intrinsic by Gauss. Defining curvature in the general Riemannian setting requires the definition of covariant derivatives of vector fields, which we prefer not to get into so we refer to [Lee19]. In Lee's book curvature is elegantly motivated by the question of whether all manifolds are locally isometric, and the definition of an intrinsic curvature shows that even locally manifolds can fundamentally differ.

Theorem 3.1 (Gauss-Bonnet). *If E is a non-degenerate geodesic polygon in M, i.e. ∂E is made up of geodesics connecting a finite number of points and has a well defined interior, then*

$$-\int_E K = (l-2)\pi - \sum_{i=1}^{l} \alpha_i,$$

where K is the Gaussian curvature of E, l is the number of vertices of E, and α_i is the interior angle at the i-th vertex.

If M is an n-manifold with $n > 2$ we can still talk about its *sectional curvature*. Given $p \in M$ and $v \in \mathrm{T}_p M$ there exits a unique maximal geodesic parametrization γ_v such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The exponential map $\exp : \mathrm{T}_p M \to M$ is defined by $\exp_p(v) = \gamma_v(1)$. If σ is a two dimensional subspace of $\mathrm{T}_p M$ then $\exp_p \sigma$ with the inherited metric is a manifold, and the sectional curvature of M with respect to σ is defined as the curvature of $\exp_p \sigma$ at the point p.

Definition 3.2. A manifold M is a hyperbolic n-space if it is geodesically complete and has constant negative sectional curvature.

We will only study the hyperbolic spaces of sectional curvature -1, but the methods presented will have analogies in the general setting due to the similarity of hyperbolic spaces [Lee19, p. 62]. The fact that we can talk about *the* hyperbolic n-space is a consequence of the *Killing-Hopf* theorem [Lee19, p. 348], and we will denote it by $\mathbb{H}^n$ whenever the choice of model is irrelevant.

This is the modern definition of hyperbolic space, but hyperbolic manifolds do arise naturally. For example when trying to construct a "sphere" of square radius -1, as is explored in [JP97]. This more natural construction yields the hyperboloid model, which we will take as our starting point. From this we will derive two classic models of hyperbolic space, which will be central in our discretizations due to their symmetries and respective properties. We compile some well-known results, accumulated from tertiary sources over the course of research, which are used in the sequel. Whilst the results are common, the proofs and presentation are original, to the extent possible. For a thorough compilation of classical hyperbolic geometry we refer to the original work by Beltrami, Klein, and Poincaré, translated by Stillwell [Sti96].

3.1 The hyperboloid model

The hyperboloid model arises in the study of the *Minkowski bilinear form* on $\mathbb{R}^{n+1}$, which is defined as

$$B(u,v) = -u_0 v_0 + \sum_{i=1}^{n} u_i v_i.$$

If we set $F(u) := B(u,u)$ we get that $\mathrm{D}F(u)(v) = 2B(u,v)$, thus $\mathrm{D}F(u) \equiv 0 \iff u = 0$, and in particular -1 is a regular value for F. The hyperboloid is defined as the regular submanifold $F^{-1}(-1)$ restricted to the upper half space

$$\mathscr{H}^n := \{(\omega_x, x) \in (0,\infty) \times \mathbb{R}^n : \omega_x^2 = 1 + \|x\|^2\}.$$

Restricting to the upper half space lets us write ω_x as a smooth function of x, showing that $\mathscr{H}^n$ is simply connected. This also lets us identify x with (ω_x, x). We will now show that B induces a metric on $\mathscr{H}^n$.

Proposition 3.3. *For any $(\omega_x, x) \in \mathscr{H}^n$ we may identify its tangent space with $\ker \mathrm{D}F((\omega_x, x))$. Moreover, B restricted to $\ker \mathrm{D}F((\omega_x, x))$ is an inner product.*

Proof. The first part is a fundamental result in the study of regular submanifolds (see e.g. [Tu11]). Since B is linear and symmetric we only need to show that it is positive definite on $\mathrm{T}\mathscr{H}^n$ in order to show that it restricts to an inner product. Fix $(\omega_x, x) \in \mathscr{H}^n$ and let $u \in \ker \mathrm{D}F((\omega_x, x))$. We compute

$$0 = \frac{1}{2}\mathrm{D}F((\omega_x, x))(u) = -\omega_x u_0 + \sum_{i=1}^{n} x_i u_i.$$

Hence by Cauchy-Schwartz

$$\omega_x^2 u_0^2 \leq \|x\|^2 \sum_{i=1}^{n} u_i^2. \tag{3}$$

In particular

$$-u_0^2 \geq \frac{-\|x\|^2}{\omega_x^2} \sum_{i=1}^{n} u_i^2 = \Big(\frac{1}{\omega_x^2} - 1\Big) \sum_{i=1}^{n} u_i^2,$$

and it follows immediately that

$$B(u,u) = -u_0^2 + \sum_{i=1}^{n} u_i^2 \geq \frac{1}{\omega_x^2} \sum_{i=1}^{n} u_i^2 \geq 0.$$

We also see that if $B(u,u) = 0$ then $\sum_{i=1}^{n} u_i^2 = 0$, and by Eq. (3) we must have that $u_0 = 0$. □

Lemma 3.4. *For any $\alpha \in \mathbb{R}$ the $(n+1) \times (n+1)$ matrix given by*

$$T^\alpha = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & \cdots \\ \sinh\alpha & \cosh\alpha & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

induces an isometry of $\mathscr{H}^n$ by $\tau : (\omega_x, x) \mapsto T^\alpha(\omega_x, x)$.

Proof. Its simple to check that T^α preserves the Minkowski form, i.e.

$$B(T^\alpha u, T^\alpha v) = B(u,v) \tag{4}$$

for all $u, v \in \mathbb{R}^{n+1}$, and so $F(u) = F(T^\alpha u)$. Moreover, on $\mathscr{H}^n$

$$\omega_x \cosh\alpha + x_1 \sinh\alpha \geq |x_1| \cosh\alpha + x_1 \sinh\alpha \geq 0,$$

so T^α maps $\mathscr{H}^n$ to the upper half plane. Then T^α maps $\mathscr{H}^n \to \mathscr{H}^n$, and since T^α is invertible it is a diffeomorphism. As a function on $\ker \mathrm{D}F$ we have $\tau_\star \equiv \mathrm{D}\tau = (T^\alpha)^\mathsf{T} = T^\alpha$, and so *Eq.* (4) shows that τ is an isometry. □

Lemma 3.5. *For any $R' \in O(n)$ the matrix given by*

$$R = \begin{bmatrix} 1 & 0 \\ 0 & R' \end{bmatrix}$$

induces an isometry of $\mathscr{H}^n$ by $\rho : (\omega_x, x) \mapsto R(\omega_x, x) = (\omega_x, R'x)$.

Proof. Again, we check that $B(Ru, Rv) = B(u,v)$ for all $u, v \in \mathbb{R}^{n+1}$, and that R maps $\mathscr{H}^n$ to the upper half plane. It then follows from the same argument as in Lemma 3.4. □

An easy extension to Lemma 3.4 is to swap the i-th row and column of T^α to obtain a family of isometries $\tau^{\alpha,i}$. We will refer to these as *translation isometries.* If we fix $(\omega_x, x) \in \mathscr{H}^n$ and consider $\tau^{\alpha,i}x$ as a function of α then it has range $(x_1, \cdots, x_{i-1}) \times \mathbb{R} \times (x_{i+1}, \cdots, x_n)$, and so there exists by composition an isometry which maps $x \mapsto 0$. Moreover, for any 2-dimensional subspace $\sigma \subset \mathrm{T}_0\mathscr{H}^n$, where $\mathrm{T}_x\mathscr{H}^n$ is the tangent space at (ω_x, x), there exists by Lemma 3.5 an isometry which maps $\sigma \mapsto \operatorname{span}\{e^1, e^2\}$. Since sectional curvature is intrinsic it follows that $\mathscr{H}^n$ must have constant sectional curvature.

Proposition 3.6. *The curve $\gamma : \mathbb{R} \to \mathscr{H}^n$ given by $\gamma_0(t) = \cosh t$, $\gamma_1(t) = \sinh t$, and $\gamma_i \equiv 0$ for $i \geq 2$, is a geodesic parametrization. Moreover, $t \mapsto \gamma(t)$ and $t \mapsto \gamma(-t)$ are the only maximal geodesic parametrizations which intersect $(1,0)$ and $(\sqrt{1+s^2}, s, 0, \cdots)$ for all $s \neq 0$.*

Proof. Given a point $(\omega_x, x) \in \mathscr{H}^n$ and $u \in \mathbb{R}^{n+1}$ there exists a unique $\Pi_x u \in \mathrm{T}_x\mathscr{H}^n$ which satisfies $B(u - \Pi_x u, v) = 0$ for all $v \in \mathrm{T}_x\mathscr{H}^n$. It is straight forward to check that $\Pi_\gamma \gamma''$ corresponds to the covariant derivative of γ'. We compute $\gamma'' = \gamma$, thus $B(\gamma'', v) = 0$ for all $v \in \mathrm{T}_x\mathscr{H}$, and we get that $\Pi_x \gamma'' = 0$. It is then a common result of differential geometry that γ is a geodesic, and since $B(\gamma', \gamma') \equiv 1$ it is a geodesic parametrization.

The set of maximal geodesic parametrizations which intersect a point can be indexed by the unit ball of the tangent space at that point. Using Lemma 3.5 and that fact that geodesics are intrinsic we may describe the set of maximal geodesic parametrizations which intersect $(1,0)$, only two of which intersect $(\sqrt{1+s^2}, s, 0, \cdots)$. □

Corollary 3.7. *Using Lemma 3.5, Lemma 3.4, and the fact that geodesics are intrinsic:*

i. For any two points $p, p' \in \mathscr{H}^n$ there is a unique maximal geodesic connecting them.

ii. A curve in $\mathscr{H}^n$ is a maximal geodesic iff it is the intersection of an n-dimensional subspace of $\mathbb{R}^{n+1}$ with $\mathscr{H}^n$.

iii. $\exp_{(1,0)}(\operatorname{span}\{e^1, e^2\})$ *is isometric to* $\mathscr{H}^2$.

iv. $\mathscr{H}^n$ *is geodesically complete.*

Corollary 3.8. *The hyperbolic distance between* (ω_x, x), $(\omega_y, y) \in \mathscr{H}^n$ *is given by*

$$\cosh^{-1}(-B[(\omega_x, x), (\omega_y, y)]).$$

Proof. If $x = 0$ then we use the geodesic parametrization from Proposition 3.6 to find that the hyperbolic distance to y is $\cosh^{-1}\omega_y = \cosh^{-1}(-B[(\omega_x, x), (\omega_y, y)])$. If $x \neq 0$ then we can find a translation isometry which maps $x \mapsto 0$, and the proposition follows from the fact that B is invariant under translation isometries.

□

Theorem 3.9. *The sectional curvature of* $\mathscr{H}^n$ *is* -1 *everywhere.*

Proof. Since the sectional curvature of $\mathscr{H}^n$ is constant it suffices to show that the curvature at $(1,0)$ of the section $\operatorname{span}\{e^1, e^2\}$ is -1. By Corollary 3.7 it suffices to show it only for $\mathscr{H}^2$. The strategy will be to approximate the area of a right isosceles triangle as the space around gets more and more flat (see Fig. 1).

Consider the geodesic curves $p(t) = (\cosh t, \sinh t, 0)$ and $q(t) = (\cosh t, 0, \sinh t)$. By Corollary 3.7 the geodesic connecting $p(t)$ and $q(t)$ is given by the projection through the origin of $\tilde{r}(s;t) := (1-s)p + sq$ onto $\mathscr{H}^2$. It's easy to verify that this projection is explicitly given by

$$r(s) = \frac{(1-s)p + sq}{\sqrt{1 + 2s(1-s)\sinh^2 t}}.$$

Then

$$r'(s) = \frac{(2s-1)(\sinh^2 t)r(s)}{1 + 2s(1-s)\sinh^2 t} + \frac{-p+q}{\sqrt{1 + 2s(1-s)\sinh^2 t}},$$

and in particular

$$\begin{aligned} r'(0) &= -(\sinh^2 t)p - p + q \\ &= (-\sinh^2 t \cosh t, (-\sinh^2 t - 1)\sinh t, \sinh t) \\ &= \cosh t \sinh t \underbrace{\left(-\sinh t, -\cosh t, \frac{1}{\cosh t}\right)}_{=:\,\theta(t)}. \end{aligned}$$

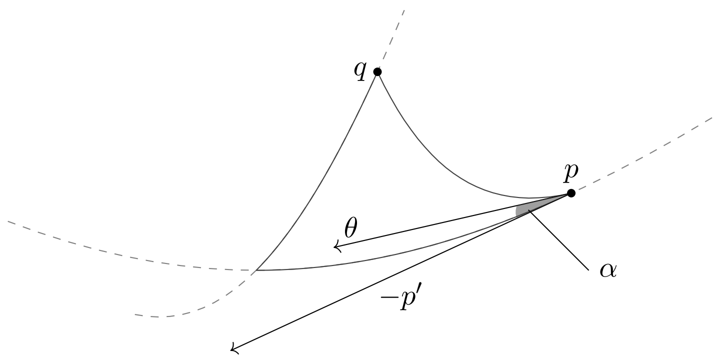


Figure 1: Hyperbolic triangle

By construction we have that $\theta \in T_p\mathscr{H}^n$. The angle between θ and $-p'$ is given by $\cos\alpha = B(-p',\theta)/\sqrt{B(\theta,\theta)}$, and this angle coincides with the other non-right angle. We compute

$$B(\theta,\theta) = 1 + \frac{1}{\cosh^2 t} = \frac{\cosh^2 t + 1}{\cosh^2 t}, \quad B(-p',\theta) = 1,$$

thus

$$\begin{aligned}\alpha &= \cos^{-1}\frac{\cosh t}{\sqrt{1+\cosh^2 t}} \\ &= \cot^{-1}(\cosh t) \\ &= \frac{\pi}{4} - \frac{t^2}{4} + O(t^4).\end{aligned}$$

By Gauss-Bonnet $\int_T -K = \pi/2 - 2\alpha$, where T is the triangle with vertices $0, p, q$. As $t \to 0$ the area of the triangle approaches $t^2/2$, and since K is constant we get that

$$-K = \lim_{t\to 0}\frac{2(\pi/2 - 2\alpha)}{t^2} = \lim_{t\to 0}\frac{t^2 + O(t^4)}{t^2} = 1.$$

□

3.2 The Poincaré model

The Poincaré model is obtained via stereographic projection onto the plane $\{(0,x) : x \in \mathbb{R}^n\}$ through $(-1,0)$ as follows:

$$\begin{aligned}(\omega_x, x) &= (-1,0) + (1+\omega_x, x) \\ &\mapsto (-1,0) + (1,\, x/(1+\omega_x)).\end{aligned}$$

This map yields a diffeomorphism of $\mathscr{H}^n$ with the unit disk in $\mathbb{R}^{n+1}$, and by adopting the inherited metric we get a new model for hyperbolic n-space.

Definition 3.10. The Poincaré ball/disk model, denoted $\mathbb{D}^n$, is defined as the unit ball in $\mathbb{R}^n$ with the metric

$$g_{ij} = \frac{4\delta_{ij}}{(1-\|x\|^2)^2}.$$

The canonical maps between $\mathscr{H}^n$ and $\mathbb{D}^n$ are given by

$$\Phi_{\mathscr{H}^n\to\mathbb{D}^n}((\omega_x, x)) = \frac{x}{1+\omega_x},$$

$$\Phi_{\mathbb{D}^n\to\mathscr{H}^n}(x) = \frac{(1+\|x\|^2, 2x)}{1-\|x\|^2},$$

and the metric on $\mathbb{D}^n$ is defined pointwise so that the following proposition holds.

Proposition 3.11. *$\Phi_{\mathbb{D}^n\to\mathscr{H}^n}$ is an isometry.*

Proof. Fix $x \in \mathbb{D}^n$. For sufficiently small t let $\gamma(t) = x + te^i$ and denote $\psi = \Phi_{\mathbb{D}^n\to\mathscr{H}^n}\circ\gamma$. We compute

$$\psi(t) = \frac{(1 + t^2 + 2tx_i + \|x\|^2,\, 2x + 2te^i)}{1 - t^2 - 2tx_i - \|x\|^2},$$

$$\psi'(t) = \frac{(2t+2x_i)\psi(t)}{1 - t^2 - 2tx_i - \|x\|^2} + \frac{(2t + 2x_i,\, 2e^i)}{1 - t^2 - 2tx_i - \|x\|^2},$$

$$(\Phi_{\mathbb{D}^n\to\mathscr{H}^n})_\star(e^i) = \psi'(0) = \frac{2x_i(1+\|x\|^2,\, 2x)}{(1-\|x\|^2)^2} + \frac{(2x_i,\, 2e^i)}{1-\|x\|^2}$$
$$= \frac{2}{1-\|x\|^2}\Big(\frac{2x_i}{1-\|x\|^2},\, \frac{2x_i x}{1-\|x\|^2} + e^i\Big).$$

Denote $\varepsilon^i = (\Phi_{\mathbb{D}^n\to\mathscr{H}^n})_\star(e^i)$. Then

$$B(\varepsilon^i, \varepsilon^j) = \frac{4}{(1-\|x\|^2)^2}\Big(\underbrace{\frac{-4x_ix_j}{(1-\|x\|^2)^2} + \frac{4x_ix_j\|x\|^2}{(1-\|x\|^2)^2} + \frac{4x_ix_j}{1-\|x\|^2}}_{=0} + \delta_{ij}\Big).$$

□

Proposition 3.12. *The hyperbolic distance between $x, y \in \mathbb{D}^n$ is given by*

$$\cosh^{-1}(-B(\Phi_{\mathbb{D}^n\to\mathscr{H}^n}(x), \Phi_{\mathbb{D}^n\to\mathscr{H}^n}(y)) = \cosh^{-1}\Big[1 + 2\frac{\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\Big].$$

In particular the hyperbolic distance from x to the origin is

$$\cosh^{-1}\frac{1+\|x\|^2}{1-\|x\|^2} = 2\tanh^{-1}\|x\|.$$

Proposition 3.13. *The inclusion $\mathbb{D}^n \to \mathbb{R}^n$ is conformal since at any point $p \in \mathbb{D}^n$*

$$\frac{\langle u, v\rangle}{\|u\|\|v\|} = \frac{\langle u, v\rangle_0}{\|u\|_0\|v\|_0} \qquad \forall u, v \in \mathrm{T_p}\mathbb{D}^\mathrm{n} \simeq \mathbb{R}^\mathrm{n}.$$

Remark 3.14. When $n = 2$ the Poincaré metric satisfies $g_{ij} = \sqrt{|g|}\,\delta_{ij}$, in which case Eq. (1) yields

$$\Delta f = \frac{1}{\sqrt{|g|}}\Delta_0 f \tag{5}$$

for all $f \in C^\infty(\mathbb{D}^2)$.

3.3 The Klein model

Similar to the Poincaré model, the Klein model is obtained via stereographic projection. This time we project onto the plane $\{(1, x) : x \in \mathbb{R}^n\}$ through $(0, 0)$ as follows:

$$(\omega_x, x) \mapsto (1, x/\omega_x),$$

which again yields a diffeomorphism between $\mathscr{H}^n$ and the unit disk in $\mathbb{R}^{n+1}$.

Definition 3.15. The Klein model, denoted $\mathbb{K}^n$, is defined as the unit ball in $\mathbb{R}^n$ with the metric

$$g_{ij} = \frac{\delta_{ij}}{1-\|x\|^2} + \frac{x_ix_j}{(1-\|x\|^2)^2}.$$

The canonical maps between $\mathscr{H}^n$ and $\mathbb{K}^n$ are

$$\Phi_{\mathscr{H}^n\to\mathbb{K}^n}((\omega_x, x)) = \frac{x}{\omega_x},$$

$$\Phi_{\mathbb{K}^n\to\mathscr{H}^n}(x) = \frac{(1, x)}{\sqrt{1-\|x\|^2}}.$$

Proposition 3.16. *$\Phi_{\mathbb{K}^n \to \mathscr{H}^n}$ is an isometry.*

Proof. Similar to that of Proposition 3.11. □

Proposition 3.17. *A curve in $\mathbb{K}^n$ is geodesic iff it is geodesic in $\mathbb{R}^n$. I.e. hyperbolic geodesics embedded in $\mathbb{K}^n$ are straight lines and vice versa.*

Proof. Let $\gamma \subset \mathbb{K}^n$ be a maximal geodesic in $\mathbb{K}^n$. By Corollary 3.7 there exists an n-dimensional subspace $S \subset \mathbb{R}^{n+1}$ such that $\Phi_{\mathbb{K}^n \to \mathscr{H}^n}(\gamma) = \mathscr{H}^n \cap S$. Then $\gamma \subset \Phi_{\mathscr{H}^n \to \mathbb{K}^n}(\{s \in S : s_0 > 0\})$, which is a geodesic in $\mathbb{R}^n$.

Conversely, if $\gamma \subset \mathbb{R}^n$ is a maximal geodesic in $\mathbb{R}^n$ (a straight line) there exists an n-dimensional subspace $S \subset \mathbb{R}^{n+1}$ such that $(1, \gamma) \subset S$. Then $\gamma = \Phi_{\mathscr{H}^n \to \mathbb{K}^n}(S)$, and in particular $\gamma \cap \mathbb{K}^n = \Phi_{\mathscr{H}^n \to \mathbb{K}^n}(S \cap \mathscr{H}^n)$, which is a geodesic in $\mathbb{K}^n$ by Corollary 3.7. □

Proposition 3.18. *The hyperbolic distance between $x, y \in \mathbb{K}^n$ is given by*

$$\cosh^{-1}(-B(\Phi_{\mathbb{K}^n \to \mathscr{H}^n}(x), \Phi_{\mathbb{K}^n \to \mathscr{H}^n}(y)) = \cosh^{-1} \frac{1 - \sum x_i y_i}{\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}}.$$

In particular the hyperbolic distance from x to the origin is

$$\cosh^{-1} \frac{1}{\sqrt{1 - \|x\|^2}} = \tanh^{-1}\|x\|.$$

We point out that by composition we get a canonical isometry between $\mathbb{D}^n$ and $\mathbb{K}^n$:

$$\Phi_{\mathbb{K}^n \to \mathbb{D}^n}(x) = \frac{x}{1 + \sqrt{1 - \|x\|^2}},$$

$$\Phi_{\mathbb{D}^n \to \mathbb{K}^n}(x) = \frac{2x}{1 + \|x\|^2}.$$

Translation and rotation isometries can be given explicitly in both models, but in applications it is simplest to first map to $\mathscr{H}^n$, apply the isometry, and then map back. We summarize the metrics of the Poincaré and Klein models in Table 1, which we will refer to later.

	Poincaré disk	Klein disk
g_{ij}	$4\kappa^2\delta_{ij}$	$\kappa\delta_{ij} + \kappa^2 x_i x_j$
g^{ij}	$\delta_{ij}/(4\kappa^2)$	$(\delta_{ij} - x_i x_j)/\kappa$
$\sqrt{\|g\|}$	$(2\kappa)^n$	$\kappa^{(n+1)/2}$

where $\kappa = (1 - \|x\|^2)^{-1}$.

Table 1: Computational reference

3.4 Sobolev spaces

In order to justify the Galerkin method we will need to construct a Sobolev space, specifically that which admits an inner product and first-order weak derivatives. The theory of Sobolev spaces is vast, even more so in the Riemannian setting. We will only present a narrow slice, which will suffice for our purposes. For an introduction to classic Sobolev theory we refer to [Eva10], and for a detailed overview in the Riemannian setting we refer to [Heb96].

Definition 3.19. Given a manifold M let $C_c^\infty(M)$ denote the set of smooth functions with compact support, i.e. the set of $\phi \in C^\infty(M)$ such that the closure of $\{x \in M : \phi(x) \neq 0\}$ is compact. The H^1-norm is defined for all $u \in C_c^\infty(M)$ by

$$\|u\|_{H^1(M)}^2 := \int_M u^2 + \|du\|^2.$$

The space $H_0^1(M)$ is the completion of $C_c^\infty(M)$ with respect to $\|\cdot\|_{H^1(M)}$. When M is compact it is standard to omit the subscript 0.

Proposition 3.20 (Poincaré inequality, [BGG17]). *If M is a submanifold of $\mathbb{H}^n$ then*

$$\int_M \|du\|^2 \geq \frac{(n-1)^2}{4} \int_M u^2 \tag{6}$$

for all $u \in H_0^1(M)$.

Remark 3.21. If M is compact then all choices of metric yield equivalent H^1-norms. In particular we may cover M with a finite number of coordinate charts and use the metric inherited from $\mathbb{R}^n$ to get a Poincaré inequality (see e.g. [Heb96]).

The significance of $H_0^1(\mathbb{H}^n)$ is that it contains some almost-everywhere differentiable functions, in particular continuous functions with sharp ridges, which commonly show up in discretizations. We will also see that the H^1-norm is a natural metric for error estimates of the Galerkin method for the Laplace equation.

We may extend Δ to act on all of $H_0^1(\mathbb{H}^n)$ as follows: Let $E \subset \mathbb{H}^n$ open and with Lipschitz-continuous boundary. For any $u \in C_c^\infty(E)$ we have by Eq. (2) that

$$-\int \langle du, d\phi\rangle = \int_E (\Delta u)\phi \qquad \forall \phi \in C_c^\infty(E), \tag{7}$$

and this uniquely determines Δu. Let $H^{-1}(E)$ denote the dual space of $H_0^1(E)$ in the distributional sense, e.g. $L^2(E) \subset H^{-1}(E)$ since for any $f \in L^2(E)$ we have that $\phi \mapsto \int_E f\phi$ is a linear form on $H_0^1(E)$. Given $u \in H_0^1(E)$ we define Δu to be the distribution corresponding to $\phi \mapsto -\int_E \langle du, d\phi\rangle$, which is a generalization by Eq. (7). By Eq. (6) the map $u, v \mapsto \int -\langle du, dv\rangle$ is coercive, so Lax-Milgram gives that $\Delta : H_0^1(E) \to H^{-1}(E)$ is a bijection.

4. Finite element methods

Given $U \subset \mathbb{H}^n$ open with Lipschitz-continuous boundary and $f \in L^2(U)$ we will take as our model problem to compute $-\Delta^{-1} f$. This is more classically phrased as

$$\begin{cases} -\Delta u = f \\ u|_{\partial U} \equiv 0 \end{cases}$$

As we have just seen, this has a unique solution in $H_0^1(U)$, and the alternative formulation is to find $u \in H_0^1(U)$ such that

$$\int_U \langle du, d\phi\rangle = \int_U f\phi \qquad \forall \phi \in H_0^1(U).$$

Here the space $H_0^1(U)$ is our *test space*, but we could just as well have used $C_c^\infty(U)$ or any other dense subspace of $H_0^1(U)$. The strategy of the Galerkin method is to cut down on the test space in a clever way, using only a finite dimensional subspace.

Let V be a subspace of $H^1_0(U)$ with finite dimension N. By Lax-Milgram there exists a unique $\tilde{u} \in V$ such that

$$\int_U \langle d\tilde{u}, dv\rangle = \int_U fv \qquad \forall v \in V, \tag{8}$$

and this is often a good approximation of u. Indeed, $d\tilde{u}$ corresponds to the L^2-projection of du onto dV, so we have that

$$\int_U \|du - d\tilde{u}\|^2 \leq \int_U \|du - dv\|^2 \qquad \forall v \in V.$$

Provided $n \geq 2$, Eq. (6) yields $\|u - \tilde{u}\|^2_{H^1(U)} \leq (\frac{4}{(n-1)^2} + 1)\|u - v\|^2_{H^1_0(U)}$ for all $v \in V$. Thus the H^1-error of $\tilde{u}$ is bounded by the error of the best approximation in V.

Definition 4.1. A degree of freedom is a non-zero linear functional $V \to \mathbb{R}$. A set of degrees of freedom is said to be *unisolvent* if they are linearly independent and the number of degrees of freedom matches the dimension of V.

Let $(\sigma_i)_{i\leq N}$ be a unisolvent set of degrees of freedom for V and let $(v_i)_{i\leq N} \subset V$ be such that $\sigma_i(v_j) = \delta_{ij}$. Then $\tilde{u} = \sum_i \sigma_i(\tilde{u})v_i$ satisfies Eq. (8) iff.

$$\sum_i \sigma_i(\tilde{u}) \int_U \langle dv_i, dv_j\rangle = \int_U fv_j$$

for all $j \leq N$. Denoting $L_{ij} = \int \langle dv_i, dv_j\rangle$ and $b_j = \int fv_j$ we may write the above as $\sum_i \sigma_i(u)L_{ij} = b_j \; \forall j \leq N$, and we immediately see how Eq. (8) transforms into a problem of linear algebra. All that remains is to solve for the degrees of freedom which then let us reconstruct $\tilde{u}$. FEM refers to methods in which we obtain the test space by subdividing U and defining v_i locally, the following construction is just one example.

A simplex of dimension n is the convex hull of $n+1$ affinely independent vertices in $\mathbb{R}^n$. For our purposes, a triangulation $\mathcal{T}$ is a collection of n-dimensional simplices such that the intersection of any two simplices is the convex hull of their shared vertices. To realize our finite element discretization we will prescribe a set of *shape functions* to each simplex.

Enumerate the non-boundary vertices $(p^i)_{i\leq N}$ and let $T \in \mathcal{T}$. If p^i belongs to T we construct $v_i|_T \in C^1(T)$ such that $v_i|_T(p^j) = \delta_{ij}$ for all other p^j which meet T. If p^i does not intersect T we set $v_i|_T \equiv 0$. We say that $(v_i|_T)_{i\leq N}$ is the set of shape functions on T and $\operatorname{span}\{v_i|_T : i \leq N\}$ is the *local test space*. If we do this for all of $\mathcal{T}$ the functions v_i are defined uniquely in the interior of each simplex, and the degrees of freedom are evaluations on p^i. If we have the additional property that $v_i|_T = v_i|_{T'}$ on $T \cap T'$ for all $T, T' \in \mathcal{T}$, then the test functions are continuous and globally defined. Such shape functions yield H^1-compatible methods, in the sense that otherwise the test space V would not actually be a subset of $H^1_0(U)$. This is certainly an important quality we want our methods to have, but is not required for the implementation.

This construction yields first order methods. To construct higher order methods we may inject products of test functions into V. In practice this is often achieved by introducing additional vertices, called *nodes*, on our simplices and doing a similar construction to that above. For example, for a two dimensional simplex T let p^i, p^j be distinct vertices of T. Supposing the shape functions yield an H^1-compatible method, the product $(v_iv_j)|_T$ is zero except in the interior of the convex hull of p^i and p^j, and so it makes sense to introduce a node at $(p^i + p^j)/2$.

4.1 Polynomial shape functions

Suppose T is a simplex contained in a coordinate chart. The simplest shape functions on T are those obtained by linear interpolation between vertices, and they can be constructed explicitly as follows. Let $\tilde{T}^n$ denote the *reference simplex*

$$\{x \in \mathbb{R}^n : \sum_i x_i \leq 1,\, x_i \geq 0\ \forall i\},$$

and define functions $\Phi_i : \tilde{T}^n \to \mathbb{R}$ by $\Phi_i(x) = x_i$ for $1 \leq i \leq n$, and $\Phi_0(x) = 1 - \sum x_i$. Denote by $p^0, \cdots, p^n$ the vertices of T and define the matrix $A_{ij} = p^j_i - p^0_i$. Then the map $F : x \mapsto p^0 + Ax$ maps $\tilde{T}^n \to T$ such that $0 \mapsto p^0$ and $e^i \mapsto p^i$, and the shape functions $v_i|_T = \Phi_i \circ F^{-1}$ yield an H^1-conforming method.

Suppose we have a triangulation $\mathcal{T}$ over the coordinates of either $\mathbb{D}^n$ or $\mathbb{K}^n$. The test space of the k-order polynomial method can be given explicitly as

$$V_k = \{v \in H^1_0(U) : v|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}\}.$$

The method given by the polynomial shape functions will vary significantly depending on our choice of coordinates, however they will serve as a point of comparison due to their simplicity. Over the coordinates of $\mathbb{D}^2$ they are especially simple since then

$$\langle dv, dv' \rangle\, \mu_g = \langle dv, dv' \rangle_0\, \mu_0,$$

so the left hand side of Eq. (8) can be computed by standard methods.

4.2 Triangulations

As a starting point in our exploration of intrinsic discretizations we will compare metrics of triangulations. We wish for triangulations of uniform size, and of good quality. The size of a triangle is measured by its greatest side length, and the quality is measured by its smallest angle. These metrics immediately translate to the hyperbolic setting, and we have computed two sets of triangulations which optimize for the Euclidean and hyperbolic metric respectively. For H.1-6 we optimize for *hyperbolic simplices*. It's not terribly difficult to define hyperbolic simplices in terms of the metric alone, but in light of Proposition 3.17 we will just say that a hyperbolic simplex is a simplex over the coordinates of $\mathbb{K}^n$.

The triangulations H.1-6 are of a disk of hyperbolic radius 3. Both E.1-6 and H.1-6 were generated by the following procedure using their respective metric:

i. Choose the number of vertices and boundary points;

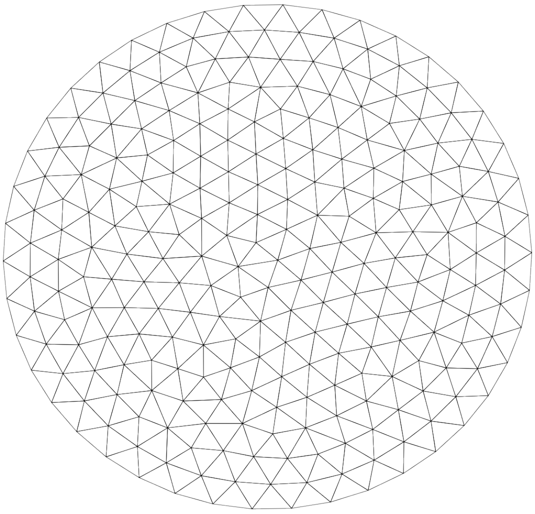
ii. set the vertices to the corresponding Fibonacci lattice (a slight modification is required to account for boundary points);

iii. compute the Delaunay triangulation[2] and perform a Voronoi iteration on non-boundary points by numerical integration. Repeat step iii. until a satisfactory triangulation is achieved.

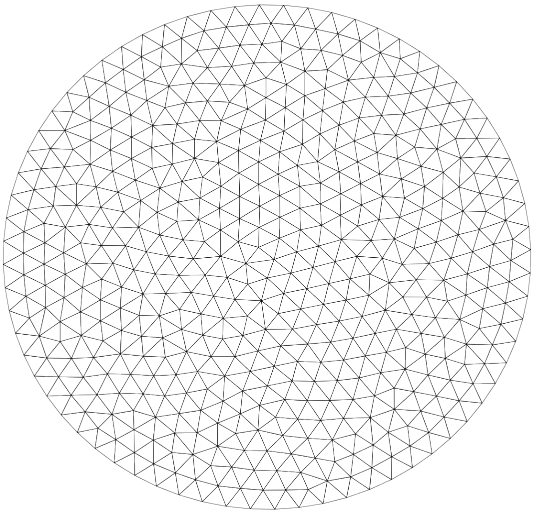
[2] Using the library https://github.com/delfrrr/delaunator-cpp/tree/master

The Fibonacci lattice is generalized as follows: Fix a point to be the origin and a geodesic going through the origin to be our primary axis. Let $N \geq 1$ be the number of vertices, and let R be the radius of the lattice. Let ρ_i be such that the volume of the ball centered at the origin with radius ρ_i is i/N the volume of the ball with radius R. The i-th vertex of the Fibonacci lattice is the point a distance ρ_i from the origin such that the geodesic from the vertex to the origin makes an angle $i\phi$ with the primary axis, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The hyperbolic Delaunay triangulation, if it exists, is equivalent to the regular Delaunay triangulation over the Poincaré model since hyperbolic circles in $\mathbb{D}^2$ are circles in $\mathbb{R}^2$. Finally, Voronoi diagrams have an obvious extension to the Riemannian setting.

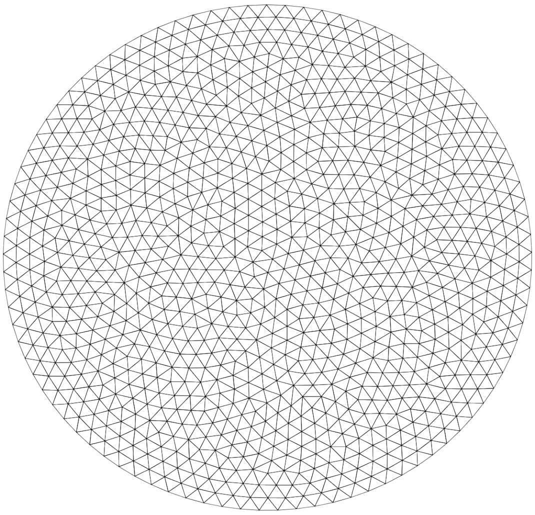
Given a number of vertices, finding a good triangulation then boils down to finding the correct number of boundary points. This is quite challenging since the above procedure is very slow. To speed it up we ended up cheating the Voronoi iteration by only computing the distance to vertices of the triangle we are integrating over.



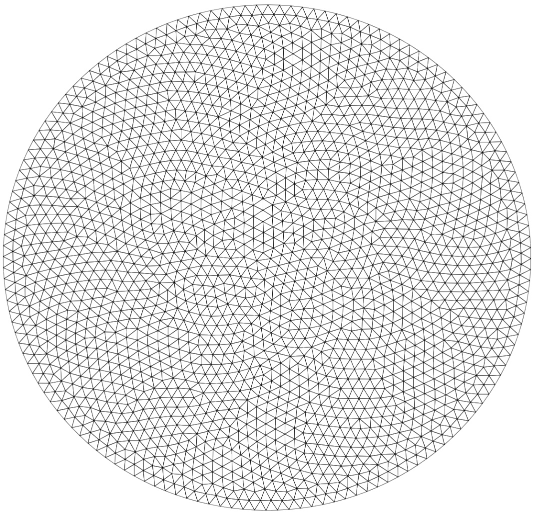
E.1



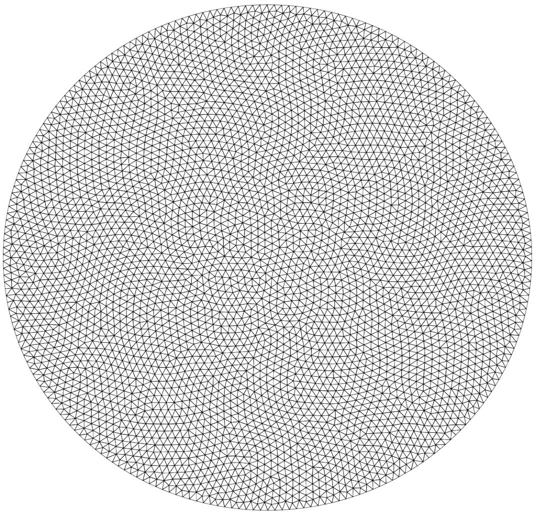
E.2



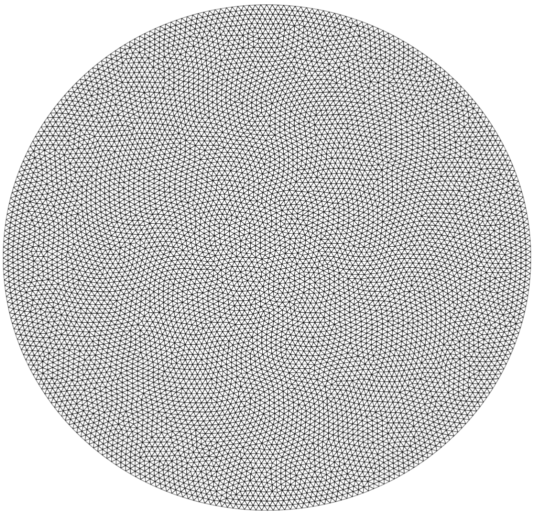
E.3



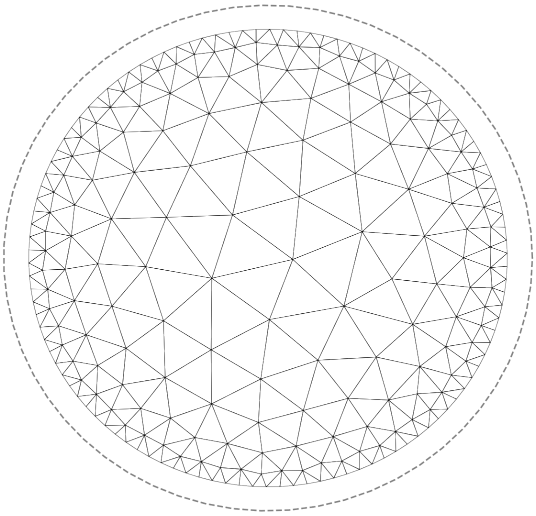
E.4



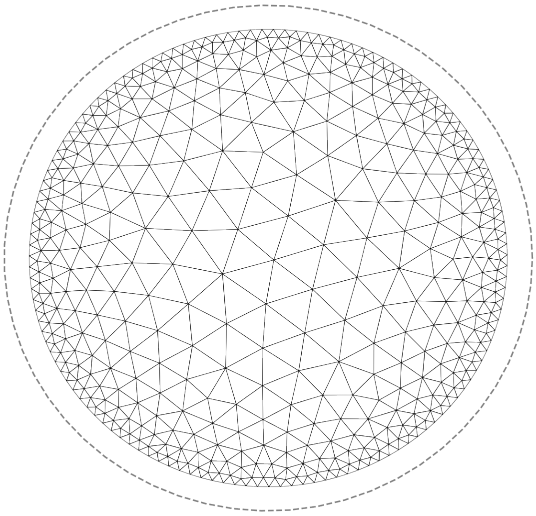
E.5



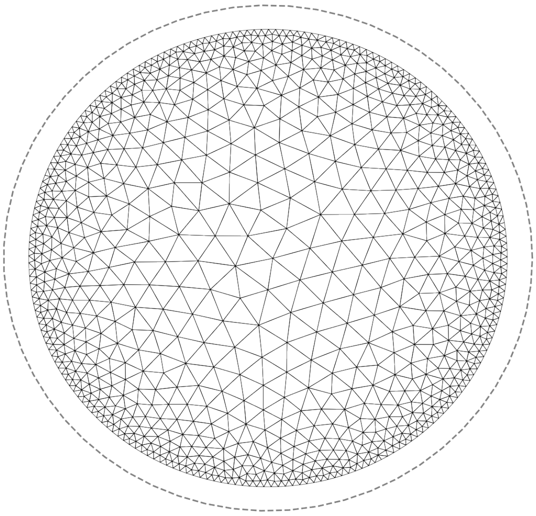
E.6



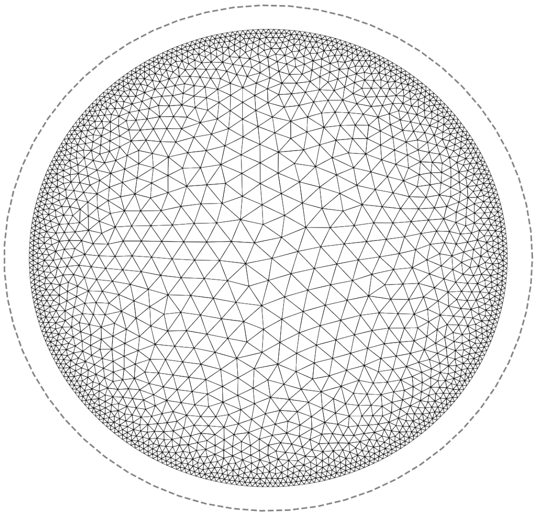
H.1



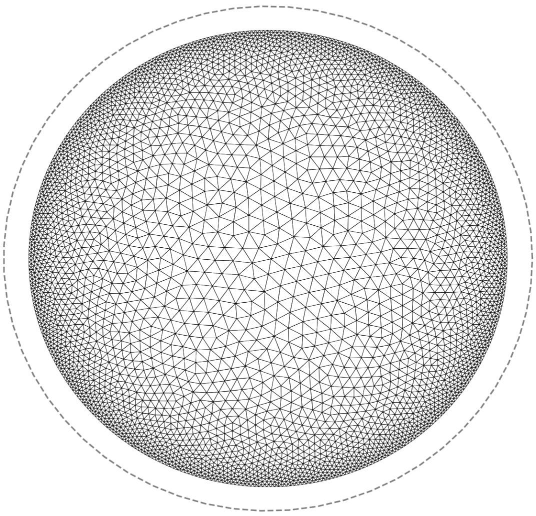
H.2



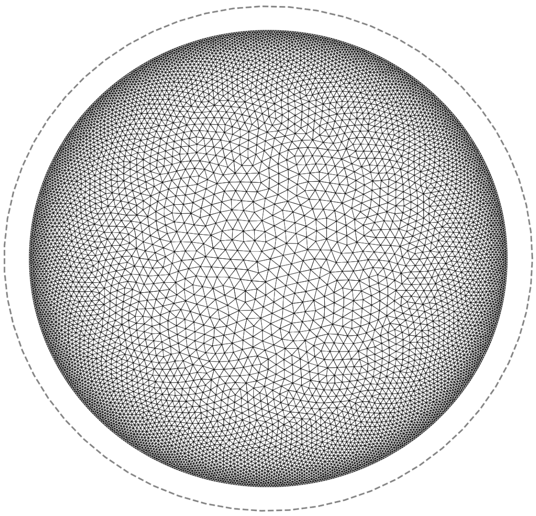
H.3



H.4



H.5



H.6

To ensure the triangulations are of high quality we have computed their size and angles in their respective metric, listed in Table 2 and Table 3. We are mainly looking for the minimum angle to be close to the mean angle and for the maximum size to be close to the mean size, which we have achieved.

	E.1	E.2	E.3	E.4	E.5	E.6
number of vertices	256	512	1024	2048	4096	8192
number of boundary vertices	42	68	89	144	230	288
minimum angle	0.749430	0.688038	0.709566	0.715066	0.667053	0.678350
mean angle	0.931138	0.936144	0.933416	0.943118	0.927955	0.933142
standard deviation of angle	0.064372	0.067722	0.060926	0.061932	0.064033	0.061677
maximum size	0.164456	0.115953	0.082284	0.056875	0.041366	0.029215
mean size	0.135922	0.094544	0.066537	0.046317	0.032912	0.022920
standard deviation of size	0.008875	0.005730	0.003963	0.002712	0.002005	0.001279

Table 2: Euclidean size and quality of E.1-6

	H.1	H.2	H.3	H.4	H.5	H.6
number of vertices	256	512	1024	2048	4096	8192
number of boundary vertices	110	131	209	288	377	610
minimum angle	0.665180	0.685812	0.618358	0.679135	0.688742	0.703381
mean angle	0.852245	0.898218	0.906763	0.912272	0.908736	0.929525
standard deviation of angle	0.070258	0.062195	0.068508	0.064946	0.071394	0.058342
maximum size	0.769777	0.515619	0.374119	0.250547	0.175876	0.123487
mean size	0.641639	0.421588	0.294200	0.204933	0.144256	0.099655
standard deviation of size	0.046496	0.032559	0.019097	0.013499	0.009875	0.005561

Table 3: Hyperbolic size and quality of H.1-6

4.3 Convergence results of intrinsic triangulations

The function $f(x) = c + \log(1 - \|x\|^2)$, defined in terms of the coordinates of $\mathbb{D}^2$, is our manifactured solution. The domain is a disk with hyperbolic radius 3, and c is such that $f \equiv 0$ on the boundary of its domain. It is straight forward to compute $-\Delta f \equiv 1$, which we solve numerically for E.1-6 and H.1-6 using the first and second order polynomial test spaces over the coordinates given by $\mathbb{D}^2$, respectively $\mathbb{K}^2$. On the y-axis is the relative L^2-error, i.e. the L^2-norm of the error divided by the L^2-norm of the solution, and on the x-axis is the number of degrees of freedom.

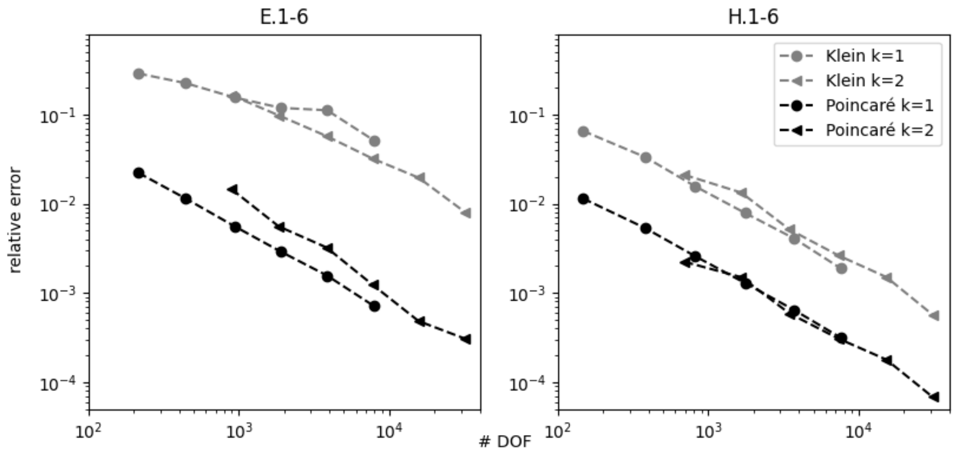


Figure 2: Convergence in standard L^2-norm

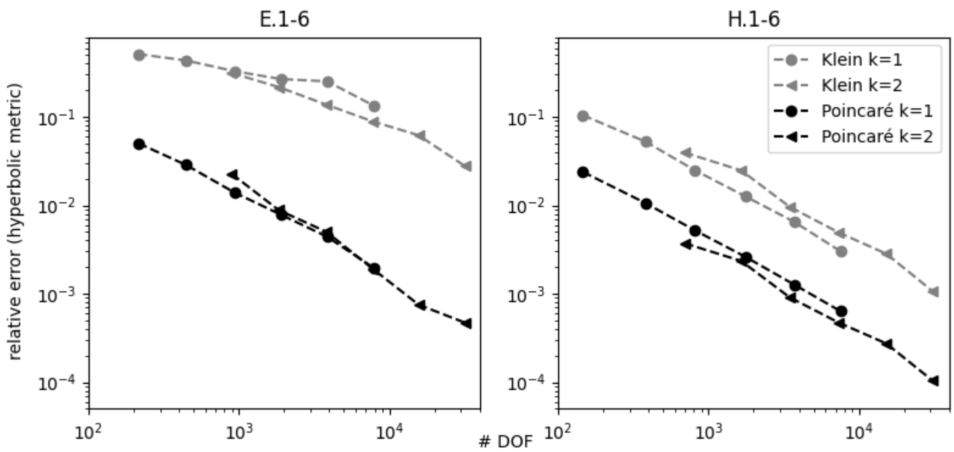


Figure 3: Convergence in hyperbolic L^2-norm

In Fig. 2 "standard L^2-norm" refers to the Euclidean L^2-norm in the coordinates corresponding to the method. As is seen in Fig. 2 and Fig. 3 the intrinsic triangulations perform significantly better. Notably E.1-6 perform terribly over the Klein model, whilst the Poincaré model admits very good convergence, perhaps due to Eq. (5). Also affecting convergence is our ability to approximate the domain. Hyperbolic simplices are small, having a bounded volume, whilst the volume of a hyperbolic ball grows exponentially with the radius. In $\mathbb{K}^2$ simplices over the coordinate chart are hyperbolic simplices so we face this issue directly.

5. Intrinsic methods

We say that an element method is intrinsic if it is coordinate-independent in the sense that its construction yields the same method independent of the underlying model. This is a bit of a meta definition as we don't have a framework for what a method or its construction is, but I think its clear what we are trying to avoid and the philosophy we are trying to bring in. Essentially, we are interested in studying FEM where the test spaces are constructed using only intrinsic properties of the underlying geometry.

Example 5.1. The polynomial test spaces do not yield intrinsic methods since

- Simplices over the coordinates of $\mathbb{D}^n$ are not hyperbolic simplices, but simplices over the coordinates of $\mathbb{K}^n$ are.
- The test spaces are not invariant under translation isometries, so the functions themselves depend on how we center our model.

Example 5.2. The shape functions constructed via harmonic extensions (see [CG16]) yield an intrinsic method since they are uniquely constructed with Δ, which is an intrinsic operator.

Given a triangle with vertices A, B, C, we will describe the shape function which evaluates to 1 at A and 0 at B and C. As we want our methods to be intrinsic the ordering of the vertices should be irrelevant, so we can just change the ordering to obtain all the shape functions. Given a point x denote by T_x the triangle spanned by x, B, C; and denote by a, b_x, c_x the side-lengths of T_x opposite x, B, C; as shown in Fig. 4.

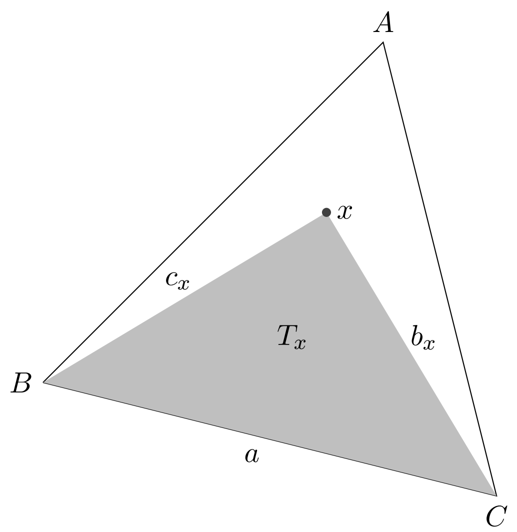


Figure 4

In order to compute side-lenghts we introduce the quantity

$$\delta_{xy} := 1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)},$$

where $x, y \in \mathbb{D}^2$. If $u, v \in \mathbb{K}^2$ (or another model) we would first need to send the points to $\mathbb{D}^2$ via an isometry before computing δ_{uv}. From Proposition 3.12 we know that the hyperbolic distance from x to y is given by $\cosh^{-1} \delta_{xy}$.

5.1 A non-existence result

Let $G(\mathbb{H}^n)$ denote the set of smooth functions which are harmonic when restricted to any geodesic, i.e. functions whose evaluation along a geodesic parametrization is affine with respect to the parameter; note that in the Euclidean setting the same construction yields the first order polynomials. This space is be a good candidate for constructing shape functions. For $n = 2$ it would immediately yield an H^1-compatible method, and it would let us explicitly compute boundary integrals even for higher order methods. Unfortunately, it turns out that $\dim G(\mathbb{H}^n) = 1$.

Proposition 5.3. *$G(\mathbb{H}^n)$ is a vector space.*

Proof. Let $f, g \in G(\mathbb{H}^n)$ and $\alpha, \beta \in \mathbb{R}$. Given a geodesic parametrization $\gamma : \mathbb{R} \to \mathbb{H}^n$ we have that $f \circ \gamma$ and $g \circ \gamma$ are affine hence

$$(\alpha f + \beta g) \circ \gamma = \alpha(f \circ \gamma) + \beta(g \circ \gamma)$$

is affine. □

Proposition 5.4. *$G(\mathbb{H}^n)$ is invariant under isometries since geodesics are intrinsic.*

Lemma 5.5. *Given $y \in (-1, 1)$ set $x(t) := \sqrt{1 - y^2} \tanh t$. Then the map*

$$\gamma : \begin{cases} \mathbb{R} \to \mathbb{K}^2 \\ t \mapsto (x(t), y) \end{cases}$$

is a geodesic parametrization. By a symmetric argument $t \mapsto (x, \sqrt{1 - x^2} \tanh t)$ is also a geodesic parametrization.

Proof. We have $\gamma'(t) = (x'(t), 0)$, and we compute

$$x'(t) = \sqrt{1 - y^2}(1 - \tanh^2 t) = \frac{1 - x(t)^2 - y^2}{\sqrt{1 - y^2}}.$$

Referring to Table 1 we get that $\langle \gamma', \gamma' \rangle = (x')^2 g_{11} = 1$. □

Theorem 5.6. $\dim G(\mathbb{K}^2) = 1$.

Proof. $G(\mathbb{K}^2)$ trivially contains the constant functions. Suppose there exists $f \in G(\mathbb{K}^2)$ such that f is non-constant. Apriori we may assume that $f(t, 0) = \tanh^{-1} t$ and $f(0, t) = 0$ for all $t \in (-1, 1)$. Indeed: Given $f \in G(\mathbb{K}^2)$ we have that $f - f(0, 0) \in G(\mathbb{K}^2)$ by Proposition 5.3, so suppose $f(0, 0) = 0$. Let $\phi : S^1 \to \mathbb{R}$ denote the evaluation of f some distance from the origin. ϕ is odd, hence there exists $s \in S^1$ such that $\phi(s) = 0$. By Proposition 5.4 we may compose f with the rotation which maps $s \mapsto (0, 1)$ to get a function in $G(\mathbb{K}^2)$ which is zero along the y-axis. Finally, suppose f is zero along the y-axis. There must exist a geodesic of the form $(x(t), y)$ on which f is non-zero, and by Proposition 5.4 we may compose f with the translation along the y-axis which maps $(x(t), y)$ to the x-axis to get a function in $G(\mathbb{K}^2)$ which is non-zero along the x-axis. After rescaling we get a function in $G(\mathbb{K}^2)$ which satisfies the above assumptions.

With these assumptions, evaluating along the geodesic parametrizations given by Lemma 5.5 reveals that for all $x, y \in \mathbb{K}^2$ and $t \in \mathbb{R}$ we have

$$f(\sqrt{1 - y^2} \tanh t, y) = \alpha(y)t,$$

$$f(x, \sqrt{1-x^2}\tanh t) = \beta(x)t + \tanh^{-1} x,$$

where $\alpha, \beta \in C^\infty((-1,1))$. By a change of variables we get the equation

$$f(x,y) = \alpha(y)\tanh^{-1}\frac{x}{\sqrt{1-y^2}} = \beta(x)\tanh^{-1}\frac{y}{\sqrt{1-x^2}} + \tanh^{-1} x,$$

which we rearrange, assuming $y \neq 0$, to get

$$\beta(x) = \frac{1}{\tanh^{-1}\frac{y}{\sqrt{1-x^2}}}\left[\alpha(y)\tanh^{-1}\frac{x}{\sqrt{1-y^2}} - \tanh^{-1} x\right].$$

Since the left hand side is a function of x alone evaluating at $\pm y$ yields the equality

$$\alpha(y)\tanh^{-1}\frac{x}{\sqrt{1-y^2}} - \tanh^{-1} x = -\alpha(-y)\tanh^{-1}\frac{x}{\sqrt{1-y^2}} + \tanh^{-1} x.$$

Then

$$(\alpha(y) + \alpha(-y)) = \frac{2\tanh^{-1} x}{\tanh^{-1}\frac{x}{\sqrt{1-y^2}}},$$

but that is a contradiction since the right hand side is not independent of x. □

Corollary 5.7. $\dim G(\mathbb{H}^n) = 1$.

Proof. By Proposition 5.4 it suffices to show that $\dim G(\mathbb{K}^n) = 1$. If $f \in G(\mathbb{K}^n)$ is non-constant let ρ be a rotation isometry such that $f \circ \rho$ is non constant on $\mathbb{K}^2 \subset \mathbb{K}^n$. Then $(f \circ \rho)|_{\mathbb{K}^2}$ is a non-constant function in $G(\mathbb{K}^2)$, contradicting Theorem 5.6. □

5.2 The volume shape function

Let $\tilde{V}(x)$ denote the hyperbolic area of T_x. The normalized volume $V(x) := \tilde{V}(x)/\tilde{V}(A)$ is a shape function, and is intrinsic. It is motivated by the fact that in the Euclidean setting the same procedure yields the polynomial basis. This construction also respects the interior of T in a unique way. A drawback of the volume shape function is that, in general, the resulting test space does not yield an H^1-conforming method.

Over the coordinates given by $\mathbb{D}^2$ we have an explicit formula of the form

$$\tilde{V}(x) = 2\tan^{-1}\frac{\sqrt{1-\delta_{BC}^2-\delta_{Bx}^2-\delta_{Cx}^2+2\delta_{BC}\delta_{Bx}\delta_{Cx}}}{1+\delta_{BC}+\delta_{Bx}+\delta_{Cx}},$$

which is derived in [Blu]. We could also compute the angles of T_x by taking advantage of Proposition 3.17 and use Gauss-Bonnet to compute $\tilde{V}$. This approach even lets us compute $\tilde{V}$ for ideal triangles.

Theorem 5.8. *V is harmonic.*

Proof. We can leverage the fact that V is intrinsic to simplify the calculation. For the triangle T_x we will denote by β the angle at B; by θ the angle between the altitude of x and the geodesic from x to B; and by w the distance from B to the foot of the altitude of x. The altitude of x splits T_x in two, and we will use the same notation with an apostrophe for the other side, as is shown in Fig. 5. Since V is intrinsic we may do our computations with the assumption that B is at the origin and C lies on the x_1-axis, as we can map them there by translation and rotation isometries. The goal will be to show that β and θ are harmonic, since then β' and θ' would be harmonic by a symmetric argument, and

$$\tilde{V} = \pi - (\theta + \theta') - \beta - \beta'$$

would be harmonic. The latter formula seems to only be applicable when T_x is acute, but it does holds in general if we ensure the sign of θ, respectively θ', is negative whenever β, respectively β', is obtuse.

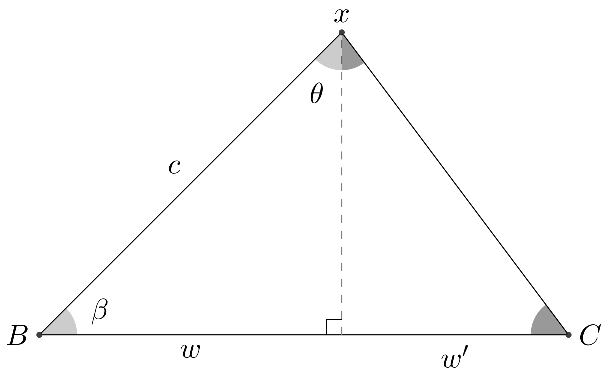


Figure 5

Suppose T_x is embedded in $\mathbb{K}^2$ such that B is the origin and C lies on the x_1-axis. As geodesics in $\mathbb{K}^2$ are straight lines, and $\mathbb{K}^2$ is conformal at the origin, we get that

$$\beta(x) = \tan^{-1}\frac{x_2}{x_1}.$$

Then

$$\mathrm{D}\beta = \frac{1}{\|x\|^2}\begin{bmatrix}-x_2\\ x_1\end{bmatrix}$$

and, referring to Table 1, we get

$$\sqrt{|g|}\,g^{-1}\mathrm{D}\theta = \frac{1}{\|x\|^2\sqrt{1-\|x\|^2}}\begin{bmatrix}1-x_1^2 & -x_1x_2\\ -x_1x_2 & 1-x_2^2\end{bmatrix}\begin{bmatrix}-x_2\\ x_1\end{bmatrix} = \frac{1}{\|x\|^2\sqrt{1-\|x\|^2}}\begin{bmatrix}-x_2\\ x_1\end{bmatrix}.$$

We compute

$$\partial_1\frac{-x_2}{\|x\|^2\sqrt{1-\|x\|^2}} = -2x_1x_2\Big[\frac{d}{d\|x\|}\frac{1}{\|x\|^2\sqrt{1-\|x\|^2}}\Big] = -\partial_2\frac{x_1}{\|x\|^2\sqrt{1-\|x\|^2}},$$

and it follows from Eq. (1) that $\Delta\beta \equiv 0$.

We now turn to θ. Since we have a right angle the hyperbolic law of sine and cosine [Hor14] simplify to

$$\sin|\theta| = \frac{\sinh w}{\sinh c}, \quad \cos|\theta| = \sin\beta\cosh w,$$

thus

$$\tan|\theta| = \frac{\tanh w}{\sin\beta\sinh c}.$$

We have that $\sin\beta = x_2/\|x\|$, and Proposition 3.18 yields $\tanh w = |x_1|$ and $\sinh c = \|x\|/\sqrt{1-\|x\|^2}$. Ensuring the aforementioned sign convention, we end up with the expression

$$\tan\theta = \frac{x_1}{x_2\sqrt{1-\|x\|^2}}.$$

From this we compute

$$\mathrm{D}\theta = \frac{1}{\|x\|^2\sqrt{1-\|x\|^2}} \begin{bmatrix} x_2(1-\frac{\|x\|^2}{1-x_1^2}) \\ -x_1 \end{bmatrix},$$

and as before

$$\begin{aligned} \sqrt{|g|}\, g^{-1}\mathrm{D}\theta &= \frac{1}{\|x\|^2(1-\|x\|^2)} \begin{bmatrix} 1-x_1^2 & -x_1x_2 \\ -x_1x_2 & 1-x_2^2 \end{bmatrix} \begin{bmatrix} x_2(1-\frac{\|x\|^2}{1-x_1^2}) \\ -x_1 \end{bmatrix} \\ &= \left[\frac{x_2}{\|x\|^2}, \; \frac{x_1x_2^2}{(1-x_1^2)(1-\|x\|^2)} - \frac{x_1}{\|x\|^2(1-\|x\|^2)}\right]. \end{aligned}$$

Finally, the same argument shows that θ is harmonic:

$$\begin{aligned} &\partial_2\Big[\frac{x_1x_2^2}{(1-x_1^2)(1-\|x\|^2)} - \frac{x_1}{\|x\|^2(1-\|x\|^2)}\Big] \\ &= \frac{2x_1x_1}{(1-x_1^2)(1-\|x\|^2)} + \frac{2x_1x_2^3}{(1-x_1^2)(1-\|x\|^2)^2} + \frac{2x_1x_2(1-2\|x\|^2)}{\|x\|^4(1-\|x\|^2)^2} \\ &= 2x_1x_2\Big[\frac{1}{(1-\|x\|^2)^2} + \frac{1-2\|x\|^2}{\|x\|^4(1-\|x\|^2)^2}\Big] \\ &= \frac{2x_1x_2}{\|x\|^4} = -\partial_1\frac{x_2}{\|x\|^2}. \end{aligned}$$

□

Corollary 5.9. *By Eq.* (2)

$$\int_\Omega \langle dV, dV\rangle = \int_{\partial\Omega} V\star dV.$$

Proposition 5.10. *Setting $V_A := V$, and denoting by V_B, V_C the analogous construction when swapping A with B, respectively C, we have that $V_A + V_B + V_C = 1$.*

5.3 The Staudtian shape function

The Staudtian is a property of geodesic triangles which appears in Euclidean, spherical and hyperbolic geometry. It relates the side-lengths of a triangle to its angles, discrete curvature, and centers [Hor14]. The hyperbolic Staudtian of T_x is given by

$$\tilde{S}(x) = \sqrt{\sinh(s_x)\sinh(s_x-a)\sinh(s_x-b_x)\sinh(s_x-c_x)},$$

where $s_x = (a+b_x+c_x)/2$ is the half-perimeter of T_x. The normalized Staudtian $S(x) := \tilde{S}(x)/\tilde{S}(A)$ is a shape function and is considered due to the following result.

Proposition 5.11. *The Staudtian shape function yields an H^1-conforming method.*

Proof. Let $x \in \partial T_A$. If x lies between B and C we have that $a = b_x + c_x$, thus $s_x - a = 0$, and so $\tilde{S}(x) = 0$. If x lies between B and A let β denote the interior angle of T_A at the B. By [Hor14, section 1.3.1, Eq. (8)] we have

$$\tilde{S}(x) = \frac{1}{2}\sin\beta\sinh a\sinh c_x,$$

hence $S(x) = \sinh c_x/\sinh c_A$, which is independent of the vertex C. A symmetric argument shows that the S is independent of B along the geodesic from C to A. □

A significant problem with the Staudtian shape function is that the resulting local test space will not contain constant functions, as they do for the polynomial and volume shape functions. This results in projections onto the test space having exacerbated discontinuities in their derivatives. In our implementation we got better results after slightly modifying the shape functions as follows: Using the notation from Proposition 5.10 we set

$$S'_A := \frac{S_A}{S_A + S_B + S_C}$$

so that $S'_A + S'_B + S'_C = 1$. Note that $S'_\cdot$ still yields an H^1-conforming method.

5.4 Convergence results of intrinsic methods

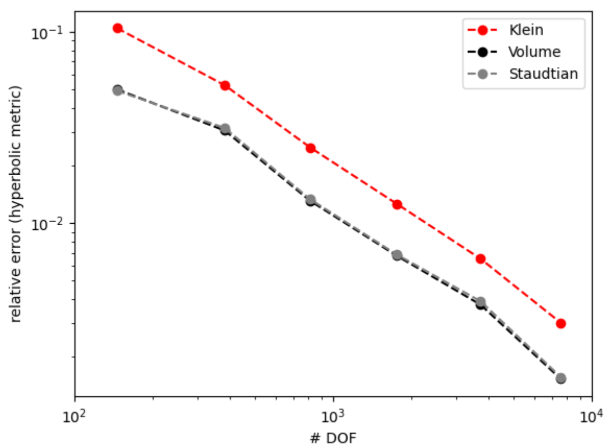


Figure 6: Convergence in hyperbolic L^2-norm

To test our methods we used the same manufactured solution as previously, and the first order polynomial method over $\mathbb{K}^2$ as our point of comparison. In the interest of keeping everything intrinsic we only used H.1-6 and the hyperbolic L^2-norm. We see in Fig. 6 that the intrinsic methods perform better than the non-intrinsic method, but not as good as polynomials over $\mathbb{D}^2$. The similarity in the error between the two distinct methods strongly suggests that a lot of the error originates with the triangulations. Perhaps we are simply failing to adequately approximate the domain, or maybe hyperbolic simplices are ill-suited for FEM.

6. Virtual element methods

The virtual element method (VEM) was introduced as an interpretation of mimetic finite differences (MFD) in the direction of FEM [Bei+13]. The result was an element method which benefits from properties of both MFD and FEM. Like MFD it naturally allows our elements to be defined on arbitrary polygons, and like FEM it naturally yields higher order methods. The most important property for our purposes is that VEM allows us to compute with a greater variety of shape functions, in particular the shape functions given by harmonic extension. The strategy of VEM is to project the shape functions onto a space that is easier to work with; in the Euclidean setting polynomials are used. The projections are constructed so that they are computable using only knowledge of the shape functions on the boundary, together with some clever choices of degrees of freedom. Our construction follows closely that presented in [Bei+23].

Let E be a geodesic polygon in $\mathbb{H}^n$ and choose a space $\mathcal{P} \subset C^1(E)$.

Definition 6.1. Given auxiliary spaces $V(\partial E) \subset H^1(\partial E)$ and $\Delta V(E) \subset C^\infty(E)$ the local test space is defined as

$$V(E) = \{v \in C^\infty(E) : \Delta v \in \Delta V(E),\, v|_{\partial E} \in V(\partial E)\}.$$

Definition 6.2. The gradient of the H^1-projection of $v \in V(E)$ is the unique element $d\Pi^1 v \in d\mathcal{P}$ which satisfies

$$\int_E \langle d\Pi^1 v,\, dp\rangle = \int_E \langle dv,\, dp\rangle \qquad \forall p \in \mathcal{P}. \tag{9}$$

Definition 6.3. The L^2-projection of $v \in L^2(E)$ is the unique element $\Pi^0 v \in \Delta\mathcal{P}$ which satisfies

$$\int_E (\Pi^0 v) p = \int_E vp \qquad \forall p \in \Delta\mathcal{P}. \tag{10}$$

The existence and uniqueness of these projections is a direct result of Lax-Milgram. They are also fairly straight forward to compute. For example, if we take a linearly independent basis $dp_1, \cdots, dp_N$ of $d\mathcal{P}$ then $d\Pi^1 v = \sum_i a_i dp_i$ satisfies Eq. (9) iff. the coefficients satisfy

$$\sum_i a_i \int_E \langle dp_i, dp_j\rangle = \int_E \langle dv, dp_j\rangle$$

for $j = 1, \cdots, N$, so we can find $a_1, \cdots, a_N$ by solving a matrix equation.

For our degrees of freedom we take a set which is unisolvent for $V(\partial E)$ together with

$$v \mapsto \int_E vp \qquad \forall p \in \Delta\mathcal{P}.$$

The DOF are chosen specifically so that the right hand side of Eq. (9) is computable:

$$\int_E \langle dv, dp\rangle = \int_{\partial E} v \star dp - \int_E v\Delta p.$$

This also clarifies why we project v onto $\Delta\mathcal{P}$ in Definition 6.3; it is the best L^2-projection we are able to compute with our limited knowledge of V. These somewhat unnatural DOF, which seem very natural when using polynomials in the Euclidean setting, introduce restrictions on the space $\mathcal{P}$. In order for the DOF to be unisolvent we must have

(U.1) $\dim \Delta V = \dim \Delta\mathcal{P}$;

(U.2) no element of $\{v \in V(E) : v|_{\partial E} \equiv 0\}$ is L^2-orthogonal to $\Delta\mathcal{P}$.

In particular if $\Delta V = \{0\}$ we must have $\Delta\mathcal{P} = \{0\}$.

The final step of VEM is to approximate $\int_E \langle dv, dv' \rangle \approx \int_E \langle d\Pi^1 v, d\Pi^1 v' \rangle$ and $\int_E vf \approx \int_E v\Pi^0 f$. This introduces a problem as

$$v, v' \mapsto \int \langle d\Pi^1 v, d\Pi^1 v' \rangle$$

is not necessarily coercive, so we don't know whether our final system will have a solution since we can't apply Lax-Milgram. To fix this we can add a *stabilizing term* to the above bilinear form so that it becomes coercive. As this was not a necessary step for our first order implementation we will omit the discussion and refer to [Bei+23] for an overview.

We must also point out that if $\Delta\mathcal{P} = \{0\}$ then $\Pi^0 \equiv 0$, so $\int_E v\Pi^0 f$ becomes a rather useless approximation In particular this occurs when V is the space of harmonic extensions, in which case [Bei+23] suggest using

$$\int_E vf \approx \frac{1}{l} \sum_{i=1}^{l} v(x^i) \int_E f, \tag{11}$$

where l is the number of vertices of E and x^i is the i-th vertex. To justify adopting this approximation for the hyperbolic metric: Let v_i be the harmonic extension such that $v_i(x^j) = \delta_{ij}$. Assuming $\int_E v_i \approx \int_E v_j$ and that E is sufficiently small we get $\int_E v_i f \approx \int_E v_j f$. Then the above approximation comes from the fact that $(v_i)_{i \le l}$ is a partition of unity, i.e. $\sum v_i \equiv 1$ hence $\sum \int_E v_i f = \int_E f$.

In order to actually compute these projections we need to have good control over integrals of functions in $\mathcal{P}$. Specifically we need to be able to compute, for all $p, p' \in \mathcal{P}$,

(I.1) $\int_E \langle dp, dp' \rangle$;

(I.2) $\int_E pp'$;

(I.3) $\int_E (\Delta p)(\Delta p')$.

This is of little restriction if we allow ourselves to use numerical integration, which we will, but for numerical accuracy it is convenient to at least know how to compute (I.1-3) over the boundary. Note that in the Euclidean metric when $\mathcal{P}$ is the space of polynomials this is a non-issue as then (I.1-3) would be integrals of polynomials over polygons.

Finally, for the spaces we are working with it is reasonable to expect

(C.1) V to be suitable for Galerkin methods;

(C.2) $d\mathcal{P} \approx dV$;

(C.3) $\Delta\mathcal{P} \approx V$;

where by "$\approx$" we indicate that we can approximate any element of one side by an element of the other in L^2-sense. These requirements are set in the hope of good convergence, though they are a bit loosely defined. The above fix for the space of harmonic extensions is an important exception to (C.3).

Remark 6.4. In the analysis of VEM an important property is that $\mathcal{P} \subset V$. This is not guaranteed in our construction, and is omitted out of necessity. If V is the space of harmonic extensions and $\mathcal{P}$ is invariant for a sufficiently diverse family of polygons, then Corollary 5.7 tells us that $\mathcal{P} \subset V \iff \mathcal{P} \subset \mathbb{R}$, which we don't want.

6.1 An implementation of VEM

Let E be a hyperbolic polygon embedded in $\mathbb{D}^2$ such that the barycenter of E is at the origin. This restriction on the origin amounts to mapping the vertices by a translation isometry before we do our computations/evaluations, and it is a nice trick to get around the fact that the choice of origin is arbitrary. Let $k \geq 1$. We set $\mathcal{P} = \mathbb{P}_k(E)$, i.e. the space of polynomials over the coordinates given by $\mathbb{D}^2$ of degree $\leq k$. As we have seen previously this space is particularly simple to work with, so it should be good to project onto. In particular we have excellent control over $\Delta\mathbb{P}_k(E)$. We set $\Delta V = \mathbb{P}_{k-2}(E)$, and set $V(\partial E)$ to be the space of continuous functions v such that for any geodesic parametrization γ of any edge of E we have that $v \circ \gamma$ is a polynomial of degree $\leq k$. This is essentially the same method as presented in [Bei+23], except that all the integrals are with respect to the hyperbolic metric. We are most interested in $k \in \{1, 2\}$, since then the local test space can be constructed intrinsically. When $k = 1$ the local test space corresponds to the space given by harmonic extensions. For the degrees of freedom on $V(\partial E)$ we take evaluations on vertices and nodes.

It should be clear that (C.1-3) hold as long as

$$M \coloneqq \sup_{x \in E} (1 - \|x\|^2)^{-2}$$

is close to 1, as this would equate to the metric being close to Euclidean. (U.1) holds since $\Delta\mathbb{P}_k = \frac{(1-\|x\|^2)^2}{4}\Delta_0\mathbb{P}_k = \frac{(1-\|x\|^2)^2}{4}\mathbb{P}_{k-2}$. (U.2) is trivial when $k = 1$ since there are no internal DOF. For $k = 2$ we note that $\phi|_{\partial E} \equiv 0$ and $\Delta\phi \equiv c$ implies that $-c\phi \geq 0$, hence if all DOF vanish

$$\int_E \langle d\phi, d\phi \rangle = -\int_E \phi\Delta\phi = -c\int_E \phi \leq -c\int_E \underbrace{M(1 - \|x\|^2)^2}_{\in \Delta\mathbb{P}_2} \phi = 0.$$

To compute the integrals we will use numerical integration over the boundary. For $p, p' \in \mathbb{P}_k$ we have that

$$\langle dp, dp' \rangle \mu_g = \underbrace{\langle dp, dp' \rangle_0}_{\in \mathbb{P}_{2(k-1)}} \mu_0,$$

and

$$(\Delta p)(\Delta p')\mu_g = \underbrace{\frac{1}{4}(1 - \|x\|^2)^2(\Delta_0 p)(\Delta_0 p')}_{\in \mathbb{P}_{2k}} \mu_0,$$

over the coordinates given by $\mathbb{D}^2$. So (I.1) and (I.3) can be computed over the boundary using anti derivation and Stokes' theorem. Computing (I.2) is slightly trickier, but doable for low values of k. Here are some possible solutions:

$$\begin{aligned}
\int_E xy^n &= 2\int_{\partial E} \frac{y^n}{1 - x^2 - y^2} dy, \\
\int_E x^2 y^n &= 2\int_{\partial E} \Big[\frac{x}{1 - x^2 - y^2} - \frac{1}{\sqrt{1 - y^2}} \tanh^{-1} \frac{x}{\sqrt{1 - y^2}}\Big] y^n dy, \\
\int_E x^3 y^n &= 2\int_{\partial E} \Big[\frac{x^2}{1 - x^2 - y^2} - \log\Big(1 - x^2 - y^2\Big)\Big] y^n dy,
\end{aligned}$$

which again are obtained via anti-derivation. For example

$$d[\frac{2y^n}{1 - x^2 - y^2} dy] = [\frac{d}{dx} \frac{2y^n}{1 - x^2 - y^2}]\mu_0 = xy^n \frac{4\mu_0}{(1 - x^2 - y^2)^2} = xy^n \mu_g.$$

6.2 Homogeneous functions

As is the theme, we will now try to alleviate some of the coordinate dependence of our VEM implementation. For a k-order VEM our previous choice of $V(\partial E)$ seems like the best intrinsic choice for our purposes, so we will focus on defining ΔV as well as the space we project onto. We will pursue a similar structure to that in [Bei+23], in the sense that we seek polynomial-like spaces $\mathcal{P}_k$ for $k = 0, 1, \cdots$, and to use $\Delta V = \mathcal{P}_{k-2}$ whilst projecting onto $\mathcal{P}_k$. These spaces should also be independent of our choice of coordinates.

Given vector spaces V, V' we recall that a function $f : V \to V'$ is said to be homogeneous of degree k if $f(sx) = s^k f(x)$ for all $x \in V$ and $s \neq 0$. The space of smooth k-degree homogeneous functions $\mathbb{R}^n \to \mathbb{R}$ is surprisingly small, in fact it leads to another way of obtaining the polynomials.

Theorem 6.5 (Euler). *$f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k if and only if $\sum x_i \partial_i f = kf$.*

Proposition 6.6.

$$\mathbb{P}_k^{\text{hom}}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : f \text{ is homogeneous of degree } k\} \tag{12}$$

Proof. [Mak] By induction on k. If $k = 0$ then $f(sx) = f(x)$ and letting $s \to 0$ we find that f is constant. Suppose Eq. (12) holds for k and let $f \in C^\infty(\mathbb{R}^n)$ be homogeneous of degree $k+1$. Differentiating both sides of $f(sx) = s^{k+1} f(x)$ yields $\partial_i f(sx) = s^k \partial_i f(x)$, thus $\partial_i f$ is homogeneous of degree k. By the induction hypothesis $\partial_i f \in \mathbb{P}_k^{\text{hom}}$, and applying Theorem 6.5 gives

$$f = \frac{1}{k+1} \sum x_i \partial_i f,$$

which shows that $f \in \mathbb{P}_{k+1}^{\text{hom}}$. □

If we set X to be the vector field over $\mathbb{R}^n$ given by $X = \sum x_i \partial_i$ then Theorem 6.5 gives

$$\mathbb{P}_k^{\text{hom}}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : df(X) = kf\}. \tag{13}$$

We can easily generalize this by allowing X to be any vector field, and this is the construction we will purse. The motivation for studying homogeneous functions came from how simple they are to integrate over polygons. In [CLS15] they use Theorem 6.5 to write the integral of a homogeneous function as an integral over the boundary of the domain. They then iteratively use the fact that the restriction of a homogeneous function to an affine subspace is homogeneous in that subspace in order to compute the integral of a homogeneous function over an n-polyhedron by its value at the vertices. We will only be able to replicate the first step, but this is sufficient for a reasonable VEM implementation in two dimensions.

Definition 6.7. Let M be a manifold. Given $X \in \mathscr{X}(M)$ and $k \in \mathbb{Z}^+$ we define

$$\text{Hom}_k^X := \{f \in C^\infty(M) : df(X) = kf\}.$$

Proposition 6.8. *If $f \in \text{Hom}_k^X$ and $g \in \text{Hom}_l^X$ then $fg \in \text{Hom}_{k+l}^X$ by the Leibniz rule.*

Immediately this is an interesting generalization since we no longer require f to be defined on a vector space. For example if we take $M = \mathbb{R}^n \setminus \{0\}$ with $X = \sum x_i \partial_i$ then Hom_k^X is infinite dimensional, indexed by the set of smooth functions on the unit sphere. If we take $M = \mathbb{R}^n$, but set $X = \sum (x_i - c_i)\partial_i$ where $c \in \mathbb{R}^n$, then Hom_k^X is the set of homogeneous polynomials in the variable $x - c$. As $x \mapsto x - c$ is an isometry this shows that these kinds of spaces may be dependent on what point we set as the origin.

Proposition 6.9. *Let $X \in \mathscr{X}(M)$ and suppose N is a manifold which is diffeomorphic to M via $\Phi : M \to N$. Let $Y = \Phi_\star X$. Then $f \in \operatorname{Hom}_k^Y \iff f \circ \Phi \in \operatorname{Hom}_k^X$.*

Proposition 6.10. *If $X \in \mathscr{X}(M)$ is such that $\operatorname{div} X \equiv c \in \mathbb{R}$ then for any Lipschitz domain $E \subset M$ and $f \in \operatorname{Hom}_k^X$*

$$\int_E f = \frac{1}{k+c} \int_{\partial E} f \star X^\flat.$$

Proof.

$$\begin{aligned} d(f\star X^\flat) &= df \wedge \star X^\flat + f d\star X^\flat \\ &= \langle df, X^\flat \rangle \mu_g + f \star^{-1} \operatorname{div} X^\flat \\ &= df(X)\mu_g + f \operatorname{div} X \mu_g \\ &= (k+c) f \mu_g. \end{aligned}$$

and the proposition follows from Stokes' theorem. □

Proposition 6.9 tells us that if X is intrinsic then so is Hom_k^X, although its not clear how we would go about defining an intrinsic vector field. For example, one could try to enforce $\operatorname{div} X \equiv 2$, $\operatorname{curl} X \equiv 0$, and $X(p) = 0$ iff. $p = p_0$ for some choice of p_0. We would then hope that somehow Hom_k^X only depends on the point p_0, but the examples below show that this is not the case.

Example 6.11. Let x be the coordinates given by $\mathbb{K}^2$ and set

$$X(x) = (1 - \|x\|^2) \begin{bmatrix} \frac{x_1}{1-x_2^2} & \frac{x_2}{1-x_1^2} \end{bmatrix}.$$

Then $\operatorname{div} X \equiv 2$, $\operatorname{curl} X \equiv 0$, and

$$\operatorname{Hom}_1^X = \operatorname{span}\left\{ x_1 \left[\frac{1 - x_2^2}{(1 - x_1^2)(1 - \|x\|^2)} \right]^{1/4}, \; x_2 \left[\frac{1 - x_1^2}{(1 - x_2^2)(1 - \|x\|^2)} \right]^{1/4} \right\}.$$

Example 6.12. Let x be the coordinates given by $\mathbb{D}^2$ and set

$$X(x) = (1 - \|x\|^2) \begin{bmatrix} x_1 & x_2 \end{bmatrix}.$$

Then $\operatorname{div} X \equiv 2$, $\operatorname{curl} X \equiv 0$, and

$$\operatorname{Hom}_1^X = \operatorname{span}\left\{ \frac{x_1}{\sqrt{1 - \|x\|^2}}, \frac{x_2}{\sqrt{1 - \|x\|^2}} \right\}.$$

If we pull back the function space of Example 6.12 to $\mathbb{K}^2$ via $\Phi_{\mathbb{K}^2 \to \mathbb{D}^2}$ we get a different space to that from Example 6.11. This shows that the conditions we have set are not strong enough to give a uniquely defined space. We will move past this since it seems unlikely that there exists a canonical choice of vector field.

We now inquire on the dimension of these spaces. As remarked before they can be infinite-dimensional, which would be inconvenient as we want to avoid arbitrarily picking a subspace. Luckily there are minimal requirements which give excellent control over the size of Hom_k^X.

Theorem 6.13. *If there exists* $\xi_1, \cdots, \xi_n \in \mathrm{Hom}_1^X$ *such that*

$$\Phi : x \mapsto (\xi_1(x), \cdots, \xi_n(x))$$

is a diffeomorphism $M \to \mathbb{R}^n$ *then* $\mathrm{Hom}_k^X = \mathbb{P}_k^{\mathrm{hom}} \circ \Phi^{-1}$. *In particular*

$$\dim \mathrm{Hom}_k^X = \dim \mathbb{P}_k^{\mathrm{hom}}.$$

Proof. Let $Y = \Phi_\star X$ and let y_i denote the i-th coordinate function on $\mathbb{R}^n$. We have that $y_i \circ \Phi = \xi_i$, thus $y_i \in \mathrm{Hom}_1^Y$ by Proposition 6.9. It's easy to check that since $dy_i(Y) = y_i$ for all i we have that $Y = \sum y_i \partial_i$. Then $\mathrm{Hom}_k^Y = \mathbb{P}_k^{\mathrm{hom}}$ by Eq. (13) and the theorem follows from Proposition 6.9. □

6.3 Constructing a complex

The goal will now be to construct polynomial-like spaces. In the process we will also derive a cochain complex; which is mostly a curiosity, though such complexes are vital in other FEM and VEM. The construction is heavily inspired by a video archive of a series of lectures on finite element exterior calculus by Douglas N. Arnold, originally held at the *2012 CBMS-NSF regional conference series in applied mathematics*.

Let M be an n-dimensional manifold. Fix a vector field X and an n-dimensional subspace $\mathcal{H}_1 \subset \mathrm{Hom}_1^X$. Let $\xi_1, \cdots, \xi_n$ be a basis for $\mathcal{H}_1$.

Definition 6.14. We set $\mathcal{H}_{-1} = \{0\}$, $\mathcal{H}_0 = \{\textit{constants}\}$, and define inductively

$$\mathcal{H}_{r+1} = \operatorname{span}\{f\xi_i : f \in \mathcal{H}_r, 1 \le i \le n\}$$

for $r \ge 0$. Note that if Hom_1^X satisfies the requirements of Theorem 6.13 then we have $\mathcal{H}_r = \mathrm{Hom}_r^X$.

Definition 6.15. In a similar fashion we define

$$\mathcal{H}_r\Lambda^{k+1} = \operatorname{span}\{\omega \wedge d\xi_i : \omega \in \mathcal{H}_r\Lambda^k, 1 \le i \le n\}.$$

Definition 6.16. The polynomial-like spaces are defined as $\mathcal{P}_r\Lambda^k := \bigoplus_{\rho=-1}^r \mathcal{H}_\rho\Lambda^k$.

It's easy to check that d maps $\mathcal{H}_r\Lambda^k \to \mathcal{H}_{r-1}\Lambda^{k+1}$, so we get a complex

$$\cdots \longrightarrow \mathcal{H}_{r+1}\Lambda^{k-1} \xrightarrow{d} \mathcal{H}_r\Lambda^k \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^{k+1} \longrightarrow \cdots . \tag{14}$$

By linearity we also get a complex of $\mathcal{P}_r\Lambda^k$, and our goal now will be to show that this complex is exact. Due to the way we defined $\mathcal{P}_r\Lambda^k$ it suffices to show that the complex in Eq. (14) is exact.

Definition 6.17. We define the operator $\kappa : \omega \mapsto \omega \lrcorner X$, where $\lrcorner$ denotes the contraction. That is, if $\omega \in \Lambda^{k+1}(M)$ then

$$\kappa\omega(V_1, \cdots, V_k) = \omega(X, V_1, \cdots, V_k),$$

where $V_i \in \mathscr{X}(M)$.

Proposition 6.18. κ *maps* $\mathcal{H}_r\Lambda^k \to \mathcal{H}_{r+1}\Lambda^{k-1}$.

Lemma 6.19. *Let* $\alpha \in \Lambda^k$ *and* $\beta \in \Lambda^l$. *Then* $\kappa(\alpha \wedge \beta) = \kappa\alpha \wedge \beta + (-1)^k \alpha \wedge \kappa\beta$.

Proposition 6.20. *κ satisfies the homotopy formula in the sense that*

$$(d\kappa + \kappa d)\omega = (k + r)\omega \tag{15}$$

for all $\omega \in \Lambda^k \mathcal{H}_r$.

Proof. Note that $\kappa d\xi_i = d\xi_i(X) = \xi_i$. If $\omega \in \mathcal{H}_r\Lambda^0$ then Eq. (15) follows from the fact that $\kappa\omega = 0$. We proceed by induction: Suppose Eq. (15) holds for k and let $\omega \wedge d\xi_i \in \mathcal{H}_r\Lambda^{k+1}$. Using Lemma 6.19 we compute

$$\begin{aligned} d\kappa(\omega \wedge d\xi_i) &= d[\kappa\omega \wedge d\xi_i + (-1)^k \omega \wedge \kappa d\xi_i] \\ &= d\kappa\omega \wedge d\xi_i + (-1)^k d\omega \wedge \xi_i + \omega \wedge d\xi_i, \end{aligned} \tag{16}$$

$$\begin{aligned} \kappa d(\omega \wedge d\xi_i) &= \kappa[d\omega \wedge d\xi_i] \\ &= \kappa d\omega \wedge d\xi_i + (-1)^{k+1} d\omega \wedge \xi_i. \end{aligned} \tag{17}$$

Putting together Eq. (16) and Eq. (17) one of the terms cancels out, and we are left with

$$\begin{aligned} (d\kappa + \kappa d)(\omega \wedge d\xi_i) &= (d\kappa + \kappa d)\omega \wedge d\xi_i + \omega \wedge d\xi_i \\ &= (k + r)\, \omega \wedge d\xi_i + \omega \wedge d\xi_i. \end{aligned}$$

By linearity it follows that Eq. (15) holds for all of $\mathcal{H}_r\Lambda^{k+1}$. $\square$

Corollary 6.21. *The complex*

$$\cdots \longrightarrow \mathcal{H}_{r+1}\Lambda^{k-1} \xrightarrow{d} \mathcal{H}_r\Lambda^k \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^{k+1} \longrightarrow \cdots$$

is exact.

Remark 6.22. This kind of structure allows us to define another common function space of FEM:

$$\mathcal{P}_r^- \Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}.$$

6.4 An intrinsic VEM

Suppose we have chosen a vector field $X \in \mathscr{X}(\mathbb{H}^2)$ and have used it to construct polynomial like spaces $\mathcal{P}_k$. To avoid arbitrary choices we suppose $\mathcal{H}_1 = \mathrm{Hom}_1^X$, i.e. $\mathcal{H}_1$ satisfies the requirements of Theorem 6.13. The final assumption will be that we know how to compute $\int_E pp'$ and $\int_E \langle dp, dp' \rangle$ for all $p, p' \in \mathcal{P}_k$ on polygonal domains E. If $\operatorname{div} X \equiv c$ we may use Proposition 6.10 to compute the first integral numerically over the boundary, in which case we will be most concerned about the second integral.

Let E be a geodesic polygon in $\mathbb{H}^2$ such that the barycenter of E is at the origin. We set $V_k(\partial E)$ to be the space of continuous functions $v \in C(\partial E)$ such that for any geodesic parametrization γ of an edge of E we have $v \circ \gamma \in \mathbb{P}_k(\mathbb{R})$, and we use the same set of DOF as before. The local test space is defined as

$$V_k(E) = \{v \in C^\infty(E) : \Delta v \in \mathcal{P}_{k-2},\ v|_{\partial E} \in V_k(\partial E)\}.$$

Letting l denote the number of edges it's easy to check that $\dim V_k(\partial E) = kl$, and it follows that $\dim V_k = kl + \dim \mathcal{P}_{k-2}$. For the internal degrees of freedom we take

$$v \mapsto \int_E vp \qquad \forall p \in \mathcal{P}_{k-2},$$

which admits a unisolvent set of DOF. Indeed, if $v \in V$ is such that all DOF vanish then

$$\int_E \langle dv, dv \rangle = \int_{\partial E} v \star dv - \int_E v \Delta v = 0.$$

This choice of DOF also lets us compute the following projection.

Definition 6.23. The L^2-projection of $v \in V_k$ is the unique element $\Pi^0_{k-2} v \in \mathcal{P}_{k-2}$ which satisfies

$$\int_E pv = \int_E p \Pi^0_{k-2} v \qquad \forall p \in \mathcal{P}_{k-2}.$$

This is a convenient choice of DOF, especially since they are immediately unisolvent, however they introduce a major challenge as we can no longer compute

$$\int_E \langle dv, dp \rangle \tag{18}$$

for $v \in V_k$ and $p \in \mathcal{P}_k$. In order to approximate Eq. (18) we will introduce an auxilary space W_k of 1-forms.

Let $W_k(\partial E)$ be the space of continuous 1-forms ω defined on ∂E such that for any geodesic parametrization γ of an edge $e \subset \partial E$ we have that $\star\omega(\gamma') \circ \gamma \in \mathbb{P}_{k-1}$. We set

$$W_k(E) = \{\omega \in C^\infty \Lambda^1(E) : d\omega \equiv 0,\ d\star\omega \in \mathcal{P}_{k-2},\ \omega|_{\partial E} \in W_k(\partial E)\}.$$

The condition $d\omega \equiv 0$ ensures control over the dimension of W_k, and it's not too hard to check that $\dim W_k \leq kl + \dim \mathcal{P}_{k-2} - 1$.

Example 6.24. A good intuition for W_1 is to think of its corresponding vector fields as divergence free vector fields whose normal component is constant along each edge. The condition $d\star\omega \equiv 0$ implies $\int_{\partial E} \star\omega = 0$, which cuts down on the dimension of W_1 by one. Fig. 7 shows how to construct a basis for W_1 on a triangle, which yields a space of dimension 2; note that $\omega_3 = -\frac{c}{b}\omega_1 - \frac{a}{b}\omega_2$.

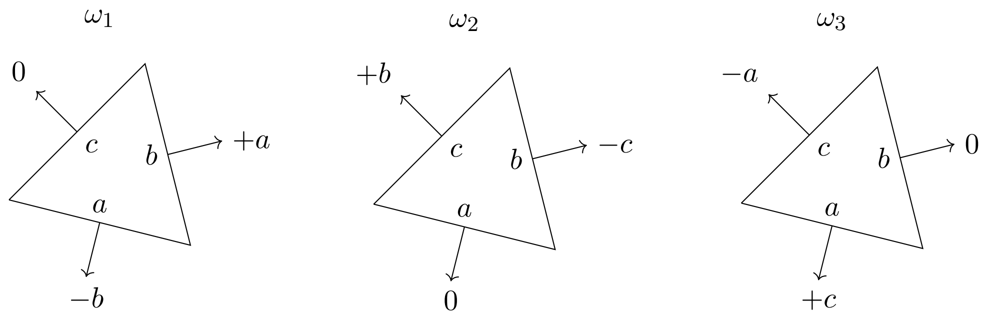


Figure 7: A basis for W_1. Here a, b, c are the side lengths of E and the arrows indiciate the value of $\star\omega(\gamma')$ when γ is a positively oriented geodesic parametrization of an edge.

The benefit of this construction is that for any $\omega \in W_k$ we can compute

$$\int_E \langle dv, \omega \rangle = \int_{\partial E} v \star\omega - \int_E v\, d\star\omega$$

for all $v \in V_k$, as well as

$$\int_E \langle dp, \omega \rangle = \int_{\partial E} p \star\omega - \int_E p\, d\star\omega$$

for all $p \in \mathcal{P}_k$. The latter lets us compute the H^1-projection of W_k onto $\mathcal{P}_k$, which in turn lets us to project dV_k onto W_k using the same strategy which enables VEM.

Definition 6.25. The gradient of the H^1-projection of $\omega \in W_k$ is the unique element $d\Pi_k^1\omega \in d\mathcal{P}_k$ which satisfies

$$\int_E \langle d\Pi_k^1\omega, dp\rangle = \int_E \langle \omega, dp\rangle \qquad \forall p \in \mathcal{P}_k. \tag{19}$$

Given $v \in V_k$ we are motivated to search for $\omega \in W_k$ which satisfies

$$\int_E \langle d\Pi_k^1\omega, d\Pi_k^1\omega'\rangle = \int_E \langle dv, \omega'\rangle$$

for all $\omega' \in W_k$, since then ω would approximate the L^2-projection of dv onto W_k. If such an ω exists its $d\Pi_k^1$-projection is unique, but existence throughout V_k fails as soon as $\ker(d\Pi_k^1 : W_k \to d\mathcal{P}_k)$ is non-empty, and in particular when $\dim W_k > \dim d\mathcal{P}_k$. To solve this we return to the idea of stabilizers. A stabilizer is a symmetric bilinear form S which is positive on $\ker(d\Pi_k^1 : W_k \to d\mathcal{P}_k)$. Taking inspiration from [Bei+23, Eq. 3.23] we will use

$$S(\omega, \omega') := h \int_{\partial E} \star\omega(\tau) \star\omega'(\tau),$$

where h is the diameter of E and τ is the positively oriented normal basis for ∂E. The factor of h is added so that S is proportional to $\int_E \langle \omega, \omega'\rangle$ with respect to h, which is important in the original method for convergence [MRR17]. Due to the geometry of large hyperbolic polygons this proportionality property won't hold in general, but it should be fine if we impose an upper bound on size and a lower bound on quality. The stabilizer enables us to define a projection $dV_k \to W_k$ as follows:

Definition 6.26. The W_k-projection of $v \in V_k$ is the unique element $\Pi_k^W v \in W_k$ which satisfies

$$\int_E \langle d\Pi_k^1\Pi_k^W v, d\Pi_k^1\omega\rangle + S((I - d\Pi_k^1)\Pi_k^W v, (I - d\Pi_k^1)\omega) = \int_E \langle dv, \omega\rangle \tag{20}$$

for all $\omega \in W_k$.

The method is then to approximate $\int_E \langle dv, dv'\rangle \approx \int_E \langle d\Pi_k^1\Pi_k^W v, d\Pi_k^1\Pi_k^W v'\rangle$ and $\int_E vf \approx \int_E v\Pi_{k-2}^0 f$, with the exception to the latter for $k = 1$ where we would use Eq. (11) instead. Alternatively we could approximate $\int_E \langle dv, dv'\rangle \approx \int_E \langle dv, \Pi_k^W v'\rangle$; note that $\int_E \langle dv, \Pi_k^W v'\rangle = \int_E \langle \Pi_k^W v, dv'\rangle$ by Eq. (20). In our tests these two approximations gave similar convergence.

6.5 Enhanced VEM

The motivation for W_k came from studying properties of $d\mathbb{P}_k$ on triangles in $\mathbb{R}^2$, but the spaces W_k tend to have greater dimension than $d\mathbb{P}_k$. It's not clear whether it would be beneficial to reduce the dimension W_k, or whether the presented construction is problematic, especially for higher order methods.

Proposition 6.27. *Let l denote the number of edges of E. Then*

$$\dim W_k = kl + \dim \mathcal{P}_{k-2} - 1. \tag{21}$$

Proof. For $k = 1$ we can give an explicit basis. Number the edges $e_1, \cdots, e_l \subset \partial E$ and for each edge let γ_i denote the positively oriented geodesic parametrization. For each $1 \le i \le l$ there exists $u \in C^\infty(E)$, unique up to a constant, such that $\star du_i(\gamma_j') \equiv \delta_{ij}/\mu(e_i)$

and $\Delta u_i \equiv 1/\mu(E)$. Given $\omega \in W_1$ there exists $u \in C^\infty(E)$ such that $\omega = du$. Since $\int_{\partial E} \star\omega = \int_E d\star\omega = 0$ the value of $\star\omega$ on e_l is determined its values on $e_1, \cdots, e_{l-1}$. It follows that

$$W_1 = \operatorname{span}_i\{du_i - du_l\}.$$

Now, suppose $k \geq 2$. For any family $(p_i)_{1 \leq i \leq l} \subset \mathbb{P}_{k-1}(\mathbb{R})$ and $q \in \mathcal{P}_{k-2} \setminus \mathbb{R}$ there exists $u \in C^\infty(E)$, unique up to a constant, such that $\star du(\gamma_i') \circ \gamma_i = p_i$ and

$$\Delta u = q + \frac{1}{\mu(E)}\Big(\int_{\partial E} \star du - \int_E q\Big).$$

Then du corresponds to an element of W_k, uniquely defined by the above DOF. Indeed, if $q \in \mathbb{R}$ and $p_e \equiv 0$ for all edges then $\int_E \Delta u = \int_{\partial E} \star du = 0$, and since Δu is constant we get

$$\int_E \langle du, du \rangle = \int_{\partial E} u \star du - \int_E u \Delta u = 0,$$

thus $du \equiv 0$. Eq. (21) then follows by counting the number of DOF. □

Corollary 6.28. $\dim W_k = \dim dV_k$.

The common way to reduce the dimension of spaces such as V_k and W_k is by imposing a set of linear equations. To demonstrate we will construct the modified trial space originally presented in [Ahm+13]. In order to do this we need to complete our definition of the projection $V_k \to \mathcal{P}_k$.

Definition 6.29. Given $v \in V_k$ the Π_k^1-projection of v is the unique element $\Pi_k^1 v \in \mathcal{P}_k$ which satisfies $d\Pi_k^1 v = d\Pi_k^1 \Pi_k^W v$ and

$$\begin{cases} \int_{\partial E} \Pi_k^1 v = \int_{\partial E} v & k = 1 \\ \int_E \Pi_k^1 v = \int_E v & k \geq 2 \end{cases}$$

The *enhanced* space is constructed by first defining a new, larger space

$$\tilde{V}_k(E) \coloneqq \{v \in C^\infty(E) : \Delta v \in \mathcal{P}_k,\ v|_{\partial E} \in V_k(\partial E)\},$$

and then using the technique described above to reduce its dimension:

$$V_k^{\mathrm{enh}}(E) \coloneqq \{v \in \tilde{V}_k(E) : \int_E (v - \Pi_k^1 v)p = 0 \quad \forall p \in \mathcal{H}_{k-1} \oplus \mathcal{H}_k\}.$$

The *enhanced* space V_k^{enh} behaves similarly to V_k in the sense that they have the same dimension and we may use the same DOF for both. The important difference is that the enhanced space allows us to compute Π_k^0 (as opposed to only Π_{k-2}^0). This is not just important for increasing the accuracy of the method, it also lets us more accurately project the solution. There is a lot of freedom in what set of constraints we choose, and one could just as well adopt the more sophisticated *serendipity* spaces (see [Bei+23]).

This leaves the question of whether it would be beneficial to do something similar for W_k. Whilst it is possible to impose integral conditions against V_k-like spaces in order to reduce the dimension to that of $d\mathcal{P}_k$, it seems hard to do so without ruining the order of accuracy. Contrary to our trial space we don't have any computational problems with W_k, so the order of accuracy is the most important property. With that in mind we believe our construction is close to optimal.

6.6 Convergence results of VEM

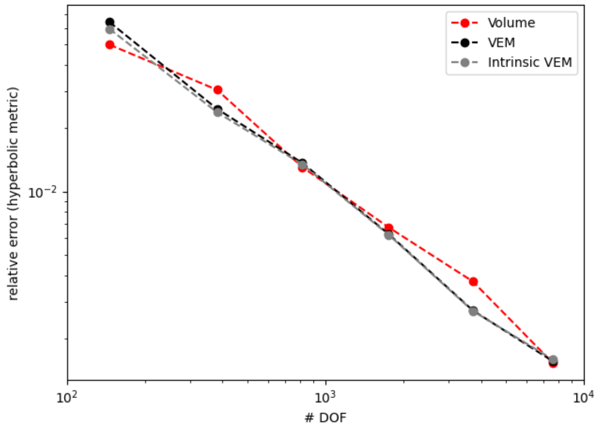


Figure 8: Convergence in hyperbolic L^2-norm

We used the same manufactured solutions as before. This time the method given by the volume shape functions is used as our point of comparison. "VEM" refers to the first order method presented in section 6.1. The intrinsic VEM was implemented using the first order polynomial-like space constructed from Example 6.12. In order to enable comparisons we needed to project our solutions onto a more tangible space, which we will quickly detail for completeness. Given a convex polygon E, a vertex $p^i \in \partial E$, and a point x in the interior of E, the maximal geodesic which connects p^i and x intersects ∂E at p^i and $q^i(x)$. We define $\tilde{\eta}_i(x) := \text{dst}(p^i, x)/\text{dst}(\text{p}^\text{i}, \text{q}^\text{i}(\text{x}))$, and $\eta_i(x) := \tilde{\eta}_i(x)/\sum_j \tilde{\eta}_j(x)$. In the end we get a partition of unity which on ∂E corresponds to the shape functions given by harmonic extensions. In retrospect it would've been more convenient to use the enhancement procedure, since then we would've had access to the L^2-projection onto the polynomial-like space.

In order to take advantage of the fact that the shape functions in VEM can be defined on arbitrary polygons we added extra vertices along the boundary. As can be seen in Fig. 8, this did not affect the convergence in a meaningful way, and standard FEM over the coordinates given by $\mathbb{D}^2$ still stands as the best performing method out of those we have implemented. When we removed the extra vertices the convergence followed more closely that of the method given by the volume shape functions.

7. Final remarks

Though we failed to improve on the standard FEM over the coordinates of $\mathbb{D}^2$, intrinsic methods may have applications when the boundary of the domain is a geodesic polygon, or when the domain is large enough that elements should be defined on distinct coordinate charts to avoid floating point error. Perhaps the problem was that hyperbolic simplices are ill-suited for FEM, it certainly would be interesting to do a construction where the functions are defined on spheres. On the contrary, optimizing for the intrinsic metrics of triangulations was beneficial, and it would be interesting to see an analytic study of convergence in terms of intrinsic properties, as well as more general algorithms for geometric meshing.

What was in my opinion the most exciting construction, the intrinsic virtual element method, was naturally motivated by the challenges of working with an underlying geometry. Even in the Euclidean setting, the idea of having an auxiliary space for computing projections opens the possibility for more variants of VEM. It remains to be seen to what extent this approach facilitates analysis.

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